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TIME DOMAIN COUPLING OF THE BOUNDARY AND FINITE ELEMENT METHODS FOR ELASTODYNAMICS

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METRIC CONVERSION FACTORS

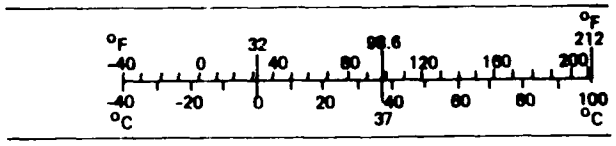
Approximate Conversions to Metric Measures

Symbol	When You Know	Multiply by	To Find	Symbol
		LENGTH		
in	inches	*2.5	centimeters	cm
ft	feet	30	centimeters	cm
yd	yards	0.9	meters	m
mi	miles	1.6	kilometers	km
		AREA		
in^2	square inches	6.5	square centimeters	cm^2
ft^2	square feet	0.09	square meters	m^2
yd^2	square yards	0.8	square meters	m^2
mi^2	square miles	2.6	square kilometers	km^2
	acres	0.4	hectares	ha
		MASS (weight)		
oz	ounces	28	grams	g
lb	pounds	0.45	kilograms	kg
	short tons (2,000 lb)	0.9	tonnes	t
		VOLUME		
tsp	teaspoons	5	milliliters	ml
Tbsp	tablespoons	15	milliliters	ml
fl oz	fluid ounces	30	milliliters	ml
c	cups	0.24	liters	l
pt	pints	0.47	liters	l
qt	quarts	0.95	liters	l
gal	gallons	3.8	liters	l
ft^3	cubic feet	0.03	cubic meters	m^3
yd^3	cubic yards	0.76	cubic meters	m^3
		TEMPERATURE (exact)		
°F	Fahrenheit temperature	5/9 (after subtracting 32)	Celsius temperature	°C

*1 in = 2.54 (exactly). For other exact conversions and more detailed tables, see NBS Misc. Publ. 286, Units of Weights and Measures, Price $2.25, SD Catalog No. C13.10:286.

Approximate Conversions from Metric Measures

Symbol	When You Know	Multiply by	To Find	Symbol
		LENGTH		
mm	millimeters	0.04	inches	in
cm	centimeters	0.4	inches	in
m	meters	3.3	feet	ft
m	meters	1.1	yards	yd
km	kilometers	0.6	miles	mi
		AREA		
cm^2	square centimeters	0.16	square inches	in^2
m^2	square meters	1.2	square yards	yd^2
km^2	square kilometers	0.4	square miles	mi^2
ha	hectares (10,000 m^2)	2.5	acres	
		MASS (weight)		
g	grams	0.035	ounces	oz
kg	kilograms	2.2	pounds	lb
t	tonnes (1,000 kg)	1.1	short tons	
		VOLUME		
ml	milliliters	0.03	fluid ounces	fl oz
l	liters	2.1	pints	pt
l	liters	1.06	quarts	qt
l	liters	0.26	gallons	gal
m^3	cubic meters	35	cubic feet	ft^3
m^3	cubic meters	1.3	cubic yards	yd^3
		TEMPERATURE (exact)		
°C	Celsius temperature	9/5 (then add 32)	Fahrenheit temperature	°F



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This study investigates the time domain coupling of the boundary and finite element methods for elastodynamics. The study is motivated by the need for a "silent boundary" in nonlinear problems with infinite domains. Of the boundary element methods (BEMs) considered, formulations based on the Stokes solution appear to be the best suited. The theoretical and numerical bases for the indirect and direct BEMs using the Stokes solution are presented. The integral equation statements of the boundary-initial value problem include a convolution of the time variable and thus appear to be computationally intense. A coupling algorithm which treats the BEM region as a nonlinear boundary condition to the finite element region is presented. The potential of the coupled approach might be realized by exploiting special properties of Stoke's solution and designing the algorithm to use the advances in computer hardware (e.g., parallel processing).

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CONTENTS

INTRODUCTION

One of the difficulties faced in the analysis of structural/geotechnical systems is numerically modeling a semi-infinite domain. The finite element method (FEM) provides an effective numerical tool for modeling complex constitutive relations which characterize soil behaviors; however, since the FEM is domain-based, it has difficulty modeling the semi-infinite domain. Simply truncating the domain at some arbitrary distance can result in the reflection of strain energy from the artificial boundary which should continue to be transmitted into the semi-infinite domain. The reflected energy can contaminate the near field region where the solution is of interest.

Considerable effort has been devoted to developing "silent" or transmitting boundaries in the frequency domain which allow for the outward radiation of energy (see e.g., Waas, 1972; Lysmer and Waas, 1972; Kausel, Roesset, and Wass, 1975; Bettess and Zienkiewicz, 1977; Karasudhi and Rajapakse, 1984). However, frequency domain solutions are limited to linear analyses.

Lysmer and Kuhlemeyer (1969) developed viscous boundaries in an early attempt to provide a "silent boundary." These boundaries are simple and can be used in both time and frequency domains. However, they act as perfect absorbers only for a limited class of problems. Cohen and Jennings (1983) give an excellent review of the different efforts made to provide a "silent boundary."

In recent years, the boundary element method (BEM) has become a strong candidate for use in the analysis of structural/geotechnical systems since it inherently satisfies the radiation boundary condition. In this study we concentrate on coupled solution approaches where boundary element methods are combined with the finite element method. The boundary element method is used to model the semi-infinite or infinite domain. In particular, we consider BEM formulations based upon the

Stokes solution -- the analytical solution for a time varying concentrated load in an infinite domain.

Objectives

The objectives of this report are to provide: (1) the necessary analytical and numerical background for time domain boundary element methods based on Stokes's solution, and (2) a coupling algorithm for combining the BEM with the FEM.

The ultimate objective of this research is to determine an effective numerical scheme for modeling nonlinear, dynamic structural/geotechnical problems which have infinite domains.

Background

The use of integral equation formulations in the analysis of transient phenomena in solids and fluids dates back over 100 years to the Helmholtz-Kirchoff integral formula, according to Manolis (1984). This formula is the mathematical description of Hygens' principle (Baker and Copson, 1939).

Though integral equation statements of wave propagation phenomena have existed for many years, their numerical approximation has occurred relatively recently. Among the early efforts were the works of Friedman and Shaw (1962) and Chen and Schweikert (1963) in acoustics, and Banaugh and Goldsmith (1963) in steady-state elastodynamics. Cruse and Rizzo (1968) and Cruse (1968) were the first to apply the BEM to transient elastodynamic problems. Their papers considered the BEM in conjunction with a Laplace transformation to solve a half-plane wave propagation problem. Other researchers considered methods using the Fourier transformation instead of the Laplace transformation (see e.g., Niwa, Kobayashi, and Azuma, 1975; Niwa, Kobayashi, and Fukui, 1976). Shaw (1985) gives an overview of many different BEM formulations in elastodynamics.

The transformation methods have been highly developed, but to address problems where the FEM region is characterized by nonlinear

behavior we must consider time domain approaches. Three time domain approaches have been pursued. They differ in one of the most basic aspects of the BEM -- the fundamental singular solution. The two simpler approaches are based on fundamental singular solutions for elliptic partial differential equations instead of the actual hyperbolic equations.

One of the approaches approximates the time derivatives by finite difference (see e.g., Brebbia and Walker, 1980). The resulting partial differential equation is solved by the boundary element method at each time step. This method requires domain integrations of previous time step displacement fields because of the finite difference approximation; thus it would appear to have limited application for problems with semi-infinite or infinite domains.

Another approach uses the Kelvin solution of elastostatic BE formulations. The main idea behind this approach is to approximate the inertia effects by expanding the displacement field throughout the domain in terms of a special set of expansion functions. This approximation is only used for the inertia terms and the expansion functions are necessarily simple allowing the domain integration to be written in terms of boundary integrations. Nardini and Brebbia (1982) first developed this approach and then followed with several applications (Nardini and Brebbia, 1983 and 1986).

Later Ahmad and Banerjee (1986) used the concepts of complementary functions and particular integrals to solve the free vibration problem in elastodynamics. Though the derivation of their approach is different than the previous work by Nardini and Brebbia, the resulting system of equations is almost identical. The reader is advised to verify the comparisons made in this later work. Both of these approaches have been applied effectively to bounded domain problems; however, I am unaware of any work which has successfully applied them to unbounded domain problems.

The most rigorous time domain approach is based upon the fundamental solution of elastodynamics -- the Stokes solution. This appears to be the best suited approach for handling infinite domains. The fundamental solution satisfies the radiation boundary conditions, and for quiescent initial conditions with no body forces, the method requires no domain integrations. However, its formulation includes a convolution of the time variable and thus appears to be computationally intense.

Attempting to provide a more efficient solution to problems with infinite domains, researchers (see e.g., Geers, 1983) have developed simplified BEM formulations known as doubly asymptotic approximations (DAA). Mita and Luco (1987) give an overview of the different BEM and coupled FEM-BEM approaches which have been used to study the dynamics of embedded foundations.

Scope

The scope of this work is limited to a discussion of time domain BEMs which are based on the use of the Stokes solution. Both analytical and numerical aspects of the formulations are presented. The analytical roots of the formulations are presented in some detail; this provides a strong background for future implementation. The emphasis is on the direct BEM formulation but the indirect BEM is also presented. The coupling algorithm treats the BEM region as a nonlinear boundary condition for the FEM region. Equilibrium and displacement compatability on the interface between the regions is satisfied in a nodal sense using an iterative scheme.

THEORY

This section presents the analytical basis for time domain BEMs based on the Stokes solution. The Nomenclature at the end of this report presents an additional explanation of some of the symbols used in this section. For additional detail the interested reader could see, for example, Eringen and Suhubi's (1974, 1975) excellent texts on elastodynamics. The later text is the main reference for this section.

The motivation for coupling the FE and BE methods to solve a boundary-initial value problem, is to apply each method to that portion of the domain for which it is best suited (see Figure 1). The finite element method in this formulation will be applied to a subdomain Ω^F which includes all portions of the domain that are nonlinear, inhomogeneous, or anisotropic. The boundary element method in this formulation will be applied to a subdomain Ω^B which will be idealized as

consisting of a homogeneous, isotropic, linear elastic material subjected to small displacements and deformations. The strength of the BEM is its inherent ability to model infinite domains. The following section presents the governing equations for Ω^B -- the equations of linear elastodynamics. For brevity, Ω^B will be denoted as Ω since the following sections only address the boundary element subdomain.

Elastodynamics Problem

Field Equations. The governing equations of motion are given by:

$$\sigma_{ji,j} + \psi_i = \rho\ddot{u}_i \tag{1a}$$

$$\sigma_{ij} = \sigma_{ji} \tag{1b}$$

and the compatibility or kinematic equations are given by:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2}$$

The BE subdomain is assumed to be an isotropic homogeneous linear elastic material. Thus the governing constitutive relations, generalized Hook's law, are given as:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} \tag{3}$$

where λ and μ are Lamé's constants expressed in terms of Young's modulus (E) and Poisson's ratio (ν) as

$$\lambda = \frac{E\nu}{(1+\nu)\ (1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \tag{4}$$

Alternatively, the equations of motion (Equation 1), compatibility relations (Equation 2), and constitutive relations (Equation 3) can be combined into a linear system of hyperbolic differential equations in terms of displacement as:

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + \psi_i = \rho\ddot{u}_i \tag{5}$$

These are the well known Navier equations of elastodynamics. These equations are often written in a slightly different form which introduces some of the complications that occur with vector hyperbolic equations. The Stokes-Helmholtz resolution theorem states that every sufficiently smooth vector field f(x,t) may be decomposed into irrotational and solenoidal parts; that is, it admits the representation:

$$\underline{f} = \underline{\nabla} f + \underline{\nabla} \times \underline{F} \tag{6}$$

where the first term is curl free and the second term is divergence free. Applying this to both the displacement and body force vector fields we can write:

$$\underline{u} = \underline{u}^P + \underline{u}^S , \qquad \underline{u}^P = \underline{\nabla}\, \phi , \qquad \underline{u}^S = \underline{\nabla} \times \underline{\chi} \tag{7a}$$

and

$$\underline{\Psi} = \underline{\nabla} f + \underline{\nabla} \times \underline{F} \tag{7b}$$

where the scalar valued function, ϕ, and the vector valued function, $\underline{\chi}$, are the Lamé potentials (Eringin and Suhubi, 1975). Substituting these relationships into Equation 5 can show that the Navier equations are satisfied if $\underline{u}^P$ and $\underline{u}^S$ satisfy:

$$C_P^2 \nabla^2 \underline{u}^P + \underline{\nabla} f = \ddot{\underline{u}}^P , \qquad C_S^2 \nabla^2 \underline{u}^S + \underline{\nabla} \times \underline{F} = \ddot{\underline{u}}^S \tag{8}$$

respectively. In the absence of body forces, these equations yield the familiar vector wave equations for irrotational and equivoluminal wave propagation, respectively. These waves propagate with velocities given by:

$$C_P = \sqrt{\frac{\lambda+2\mu}{\rho}} , \qquad C_S = \sqrt{\frac{\mu}{\rho}} \tag{9}$$

Noting that λ and μ are positive implies that $C_P > C_S$ and, thus, the rationale for the subscripts: P for 'primary' corresponding to the faster wave, and S for 'secondary' corresponding to the slower wave. In addition to irrotational and primary, the P waves are referred to as:

dilational, pressure, compressional, and longitudinal waves. In addition to equivoluminal and secondary, the S waves are referred to as: distortional, shear, and transverse waves.

Navier's equations are often written in terms of these wave speeds as:

$$C_S^2 u_{i,jj} + \left(C_P^2 - C_S^2\right) u_{j,ji} + \Psi_i = \ddot{u}_i \tag{10}$$

The equations of motion, kinematic relations, and constitutive relations govern the problem over the domain except for singular surfaces where special jump conditions must be satisfied. The wave front of a shock wave is an example of a singular surface. For additional details on singular surfaces and the corresponding jump conditions see Eringen and Suhubi (1974).

Initial Conditions. The initial conditions give the displacement and velocity field throughout the domain as:

$$u_i(\underline{x},0) = u_{oi}(\underline{x}) \qquad \underline{x} \in \Omega \tag{11a}$$

$$\dot{u}_i(\underline{x},0) = v_{oi}(\underline{x}) \qquad \underline{x} \in \Omega \tag{11b}$$

Boundary Conditions. The boundary conditions are given by:

$$u_i(\underline{x},t) = \hat{u}_i(\underline{x},t) \qquad \underline{x} \in \Gamma_U \,, \quad t \in [0,\infty) \tag{12a}$$

$$t_i(\underline{x},t) = \hat{t}_i(\underline{x},t) \qquad \underline{x} \in \Gamma_T \,, \quad t \in [0,\infty) \tag{12b}$$

where $\underline{\hat{u}}(\underline{x},t)$ and $\underline{\hat{t}}(\underline{x},t)$ are prescribed distributions of boundary displacements and tractions as a function of time. The simple notation does not imply that the boundary conditions are mutually exclusive; the fully-mixed boundary value problem is addressed.

Radiation Boundary Conditions. If the domain is unbounded, physics places constraints on the behavior of the fields at infinity. Physical reasoning suggests that if the applied loading is restricted to a finite

region, waves propagating from infinity into the interior of the domain should not exist.

Radiation boundary conditions are attributed to Sommerfeld (1949). He suggests the necessity of such a condition in the following discussion:

> "... oscillation problems (in constrast to potential problems) are not determined uniquely by their prescribed sources in the finite domain. This paradoxical result shows that the condition of vanishing at infinity is not sufficient, and that we have to replace it by a stronger condition at infinity. We call it the condition of radiation: the sources must be sources, not sinks of energy. The energy which is radiated from the sources must scatter to infinity; no energy may be radiated from infinity into the prescribed singularities of the field."

Regardless of whether we are seeking a closed form or a numerical solution, the radiation condition provides the same essential property to our solution -- uniqueness.

It can be proven (Eringen and Suhubi, 1975) that the radiation conditions of elastodynamics are direct consequences of the radiation conditions on wave equations (Equation 8). The radiation conditions of elastodynamics can be stated as:

$$\lim_{r\to\infty} r \left[\underline{t}^P + \rho C_P \left(\partial \underline{u}^P / \partial t \right) \right] = \underline{0} \tag{13a}$$

$$\lim_{r\to\infty} r \left[\underline{t}^S + \rho C_S \left(\partial \underline{u}^S / \partial t \right) \right] = \underline{0} \tag{13b}$$

where $\underline{t}^P$ and $\underline{t}^S$ are traction vectors on a sphere of radius r, due to the displacement components $\underline{u}^P$ and $\underline{u}^S$, respectively. These conditions are sufficient to guarantee that at infinity there will only be an outward flow of energy; that is, reflections are eliminated.

Integral Equation Formulations

This section presents the integral equations on which both the direct and indirect BEMs are based. Only the equations for the direct formulation are developed in detail. The development of the direct formulation in elastodynamics strongly parallels the development in elastostatics; the two major components are a fundamental singular solution and a reciprocal theorem. Kelvin's solution is replaced by Stokes's solution, and Betti's reciprocal theorem is extended to Graffi's theorem.

Elastodynamic State. Prior to presenting Stokes's solution and deriving Graffi's theorem, it is useful to have the definition of elastodynamic state as given by Eringen and Suhubi (1975, Section 5.7).

Let Ω be a spacial region with boundary Γ, and T a time interval. If $\underline{u}$ and σ are, respectively, a vector-valued and a symmetric second-order tensor-valued function defined on $\Omega \times T$, we call the ordered pairs $y = [\underline{u},\sigma]$ an **elastodynamic state** on $\Omega \times T$ with the displacement field $\underline{u}$ and stress field σ, corresponding to a body force density $\underline{\Psi}$, mass density ρ, irrotational wave speed C_P, and equivoluminal wave speed C_S, provided that:

(a) $\underline{u} \in C^{2,2}(\Omega \times T)$, $\underline{u} \in C^{1,1}(\Gamma \times T)$, $\sigma \in C^{0,0}(\Omega \times T)$,

as $\underline{\Psi} \in C^{0,0}(\Omega \times T)$, $\rho > 0$, $C_P > 2/\sqrt{3}\, C_S > 0$ (14)

(b) $\underline{u}$, σ, $\underline{\Psi}$, ρ, C_P, and C_S satisfy the governing Equations 1, 2, and 3.

The class of all elastodynamic states satisfying the above conditions is denoted by E where we write:

$$y \in E\,(\underline{\Psi},\rho,C_P,C_S;\ \Omega \times T) \tag{15}$$

When

$$T = T^{\infty} \quad \text{and} \quad \underline{u} = \underline{0} \text{ on } \Omega \times T^{-}$$

we refer to y as an **elastodynamic state of quiescent past** and write:

$$y \in E_o(\underline{\Psi}, \rho, C_P, C_S; \Omega) \tag{16}$$

See Nomenclature at end of report for a description of the notation used in the continuity conditions of (a) above.

Stokes's Solution. Stokes's problem consists of an infinite domain subjected to a concentrated load at a point $\underline{\xi}$ which is fixed in direction e_i but has an arbitrary time varying magnitude. That is, we seek the solution in an infinite domain where the body force is given as:

$$\psi_i(\underline{x},t) = f(t)\ \delta(\underline{x}-\underline{\xi})\ e_i \tag{17}$$

The solution is the fundamental singular solution of elastodynamics and is originally due to Stokes (1849) (also see Love, 1944, Section 212 or Eringen and Suhubi, 1975, Section 5.10). The displacement at a position $\underline{x}$ and time t is given as:

$$u_i(\underline{x},t) = u_{ij}(\underline{x},t;\underline{\xi}|f)\ e_j \tag{18a}$$

and

$$u_{ij}(\underline{x},t;\underline{\xi}|f) = u^P_{ij}(\underline{x},t;\underline{\xi}|f) + u^S_{ij}(x,t;\xi|f) \tag{18b}$$

$$u^P_{ij}(\underline{x},t;\underline{\xi}|f) = \frac{1}{4\pi\rho r}\left[-\left(\frac{3r_i r_j}{r^2} - \delta_{ij}\right)\int_0^{C_P^{-1}} \lambda\ f(t-\lambda r)\ d\lambda\right.$$

$$\left. + \frac{r_i r_j}{r^2 C_P^2}\ f'\left(t - \frac{r}{C_P}\right)\right| \tag{18c}$$

$$u^S_{ij}(\underline{x},t;\underline{\xi}|f) = \frac{1}{4\pi\rho r}\left[\left(\frac{3r_ir_j}{r^2} - \delta_{ij}\right)\int_0^{C_S^{-1}} \lambda\, f(t-\lambda r)\, d\lambda\right.$$

$$\left. - \frac{r_ir_j}{r^2C_S^2} f\left(t - \frac{r}{C_S}\right) + \frac{\delta_{ij}}{C_S^2} f\left(t - \frac{r}{C_S}\right)\right] \tag{18d}$$

$$r_i = x_i - \xi_i \tag{18e}$$

$$r^2 = r_ir_i \tag{18f}$$

u^P_{ij} and u^S_{ij} are the irrotational and equivoluminal parts of the Stokes tensor u_{ij}. Substituting this result into the kinematic (Equation 2) and constitutive equations (Equation 3) one can obtain the stress at $(\underline{x},t)$ as (Eringen and Suhubi, 1975):

$$\sigma_{ij}(\underline{x},t) = \sigma_{ijk}(\underline{x},t;\underline{\xi}|f)\, e_k \tag{19a}$$

where

$$\sigma_{ijk}(\underline{x},t;\underline{\xi}|f) = \sigma^P_{ijk}(\underline{x},t;\underline{\xi}|f) + \sigma^S_{ijk}(\underline{x},t;\xi|f) \tag{19b}$$

$$\sigma^P_{ijk}(\underline{x},t;\underline{\xi}|f) = \frac{1}{4\pi r^2}\left\{6\, C_S^2\left[\frac{5r_ir_jr_k}{r^3} - \frac{\delta_{ij}r_k + \delta_{ik}r_j + \delta_{jk}r_i}{r}\right]\right.$$

$$\times \int_0^{C_P^{-1}} \lambda\, f(t-\lambda r)\, d\lambda - \frac{2C_S^2}{C_P^2}\left[\frac{6r_ir_jr_k}{r^3} - \frac{\delta_{ij}r_k + \delta_{ik}r_j + \delta_{jk}r_i}{r}\right]$$

$$\times f\left(t - \frac{r}{C_P}\right) - \left(1 - 2\frac{C_S^2}{C_P^2}\right)\frac{r_k\delta_{ij}}{r}\left[f\left(t - \frac{r}{C_P}\right) + \frac{r}{C_P}\dot{f}\left(t - \frac{r}{C_P}\right)\right]$$

$$\left. - \frac{2r_ir_jr_k}{r^2}\frac{C_S^2}{C_P^3}\dot{f}\left(t - \frac{r}{C_P}\right)\right\} \tag{19c}$$

$$\sigma^{S}_{ijk}(\underline{x},t;\underline{\xi}|f) = \frac{1}{4\pi r^2}\left\{-6\,C_S^2\left[\frac{5r_ir_jr_k}{r^3} - \frac{\delta_{ij}r_k + \delta_{ik}r_j + \delta_{jk}r_i}{r}\right]\right.$$

$$\times \int_0^{C_S^{-1}} \lambda\, f(t-\lambda r)\, d\lambda + 2\left[\frac{6r_ir_jr_k}{r^3} - \frac{\delta_{ij}r_k + \delta_{ik}r_j + \delta_{jk}r_i}{r}\right]$$

$$\times f\left(t - \frac{r}{C_S}\right) - \frac{\delta_{ik}r_j + \delta_{jk}r_i}{r}\left[f\left(t - \frac{r}{C_S}\right) + \frac{r}{C_S}\,\dot{f}\left(t - \frac{r}{C_S}\right)\right]$$

$$\left. + \frac{2r_ir_jr_k}{r^2C_S}\,\dot{f}\left(t - \frac{r}{C_S}\right)\right\} \qquad (19d)$$

Note that with the above notation we can consider u_{ij} and σ_{ijk} as operators on the time variation of the concentrated load f(t). The quantities of the form (t-r/C) are referred to as retarded times; the effect of the temporal variation of the load is retarded by the time it takes a wave to travel from the source to the field point.

The elastodynamic state, which corresponds to Stokes's solution with the concentrated force acting parallel to the x_k-axis, is referred to as the **Stokes state** and denoted (Eringen and Suhubi, 1975, Section 5.10) by:

$$y_k(\underline{x},t;\underline{\xi}|f) = \left[\underline{u}_k(\underline{x},t;\underline{\xi}|f),\ \boldsymbol{\sigma}_k(\underline{x},t;\underline{\xi}|f)\right] \qquad (20a)$$

where the vectors $\underline{u}_k$ and second-order tensors $\boldsymbol{\sigma}_k$ (k=1,2,3) are:

$$\underline{u}_k = \left[u_{ik}\right], \qquad \boldsymbol{\sigma}_k = \left[\sigma_{ijk}\right] \qquad (20b)$$

In particular, we will often need to refer to a class of Stokes's states that have a quiescent past. Consider the following definition given by Wheeler and Sternberg (1968):

Let $\underline{\xi} \in \mathbf{E}_3$ and f be a twice continuously differentiable function of time that vanishes on T^-, and ρ, C_P, C_S, satisfy Equation 14. We call the state $y_k^o(\underline{x},t;\underline{\xi}|f) = [\underline{u}_k^o, \sigma_k^o]$ defined on $\mathbf{E}_{3\xi} \times T^\infty$ by Equations 20, 18, and 19 the **Stokes state of quiescent past** for a concentrated force at $\underline{\xi}$ parallel to the x_k-axis corresponding to the force function f(t) and to the material constants ρ, C_P, C_S.

If we assume Equation 18 is valid for an impulse force, i.e., $f(t) = \delta(t-\tau)$, the substitution leads to the free space Green's function:

$$G_{ij}(\underline{x},t;\underline{\xi},\tau) = u_{ij}[\underline{x},t;\underline{\xi}|\delta(t-\tau)] \tag{21a}$$

So this expression gives the i^{th} component of the displacement field at $(\underline{x},t)$ due to the j^{th} component of a concentrated impulse acting at $(\underline{\xi},\tau)$. In some contexts the Green's function is written for $\tau = 0$ as:

$$G_{ij}(\underline{x},t;\underline{\xi}) = G_{ij}(\underline{x},t;\underline{\xi},0) \tag{21b}$$

By considering the "sifting property" of the δ-function, integration of Stokes's solution gives the Green's function as:

$$\begin{aligned} G_{ij}(\underline{x},t;\underline{\xi},\tau) = \frac{t'}{4\pi\rho r^2}\Bigg\{ & \left(\frac{3r_i r_j}{r^3} - \frac{\delta_{ij}}{r}\right)\left[H\left(t' - \frac{r}{C_P}\right) - H\left(t' - \frac{r}{C_S}\right)\right] \\ & + \frac{r_i r_j}{r^2}\left[\frac{1}{C_P}\,\delta\left(t' - \frac{r}{C_P}\right) - \frac{1}{C_S}\,\delta\left(t' - \frac{r}{C_S}\right)\right] \\ & + \frac{\delta_{ij}}{C_S}\,\delta\left(t' - \frac{r}{C_S}\right)\Bigg] \end{aligned} \tag{22}$$

where $t' = t - \tau$.

The above result for the displacement field, when substituted into the kinematic and constitutive relations, gives the stress field as:

$$\sigma_{ij}(\underline{x},t) = T_{ijk}(\underline{x},t;\underline{\xi},\tau)\, e_k(\underline{\xi}) \tag{23a}$$

where

$$\begin{aligned} T_{ijk}(\underline{x},t;\underline{\xi},\tau) = \frac{1}{4\pi r^2}\Bigg\{ & -6\,\frac{C_S^2\, t'}{r^2}\left[5\,\frac{r_i r_j r_k}{r^3} - \frac{\delta_{ij}r_k + \delta_{ik}r_j + \delta_{jk}r_i}{r}\right] \\ & \times\left[H\left(t' - \frac{r}{C_P}\right) - H\left(t' - \frac{r}{C_S}\right)\right] \\ & + 2\left[6\,\frac{r_i r_j r_k}{r^3} - \frac{\delta_{ij}r_k + \delta_{ik}r_j + \delta_{jk}r_i}{r}\right] \\ & \times\left[\delta\left(t' - \frac{r}{C_S}\right) - \frac{C_S^2}{C_P^2}\,\delta\left(t' - \frac{r}{C_P}\right)\right] \\ & + 2\,\frac{r_i\, r_j\, r_k}{r^2\, C_S}\left[\dot{\delta}\left(t' - \frac{r}{C_S}\right) - \frac{C_S^3}{C_P^3}\,\dot{\delta}\left(t' - \frac{r}{C_P}\right)\right] \\ & - \frac{r_i\delta_{jk}}{r}\left(1 - 2\,\frac{C_S^2}{C_P^2}\right)\left[\delta\left(t' - \frac{r}{C_P}\right) + \frac{r}{C_P}\,\dot{\delta}\left(t' - \frac{r}{C_P}\right)\right] \\ & - \frac{\delta_{ik}r_j\ \delta_{ij}r_k}{r}\left[\delta\left(t' - \frac{r}{C_S}\right) + \frac{r}{C_S}\,\dot{\delta}\left(t' - \frac{r}{C_S}\right)\right]\Bigg\} \end{aligned} \tag{23b}$$

By Cauchy's formula one can obtain the relationship for the traction on a plane with unit normal n_i as:

$$t_i(\underline{x},t) = F_{ik}(\underline{x},t;\underline{\xi},\tau)\, e_k(\underline{\xi}) \tag{24a}$$

where

$$F_{ik}(\underline{x},t;\underline{\xi},\tau) = T_{ijk}(\underline{x},t;\underline{\xi},\tau)\, n_j(\underline{x}) \tag{24b}$$

It should be noted that many equivalent forms of Equations 22 and 23 are given in the literature. The above classical solutions to the equations of elastodynamics are an essential element to the time domain boundary element methods of this study. The second essential element is the dynamic reciprocal theorem considered in the following section.

Dynamic Reciprocal Theorem. In elastostatics the Betti-Rayleigh reciprocal work theorem provides a relationship between two distinct equilibrium states. The dynamic reciprocal theorem can be written by applying the D'Alembert principle -- including the inertia terms as a part of the body force. By this approach, Rayleigh (1873) obtained a theorem (Love, 1944, Section 121) from which Graffi's theorem (1946-1947) can be deduced; we will take this approach in the following discussion. For another proof of Graffi's theorem see, for example, Eringen and Suhubi (1975, Section 5.8).

The following proof of Graffi's theorem is included for more than the sake of completeness. A numerical method cannot be properly used, let alone be implemented, without a basic understanding of the underlying analysis. The development which follows also uses notation common to much of the literature.

Consider two distinct elastodynamic states:

$$y = [\underline{u},\sigma] \in E(\underline{\Psi},\rho,C_p,C_s;\ \Omega \times T^+) \tag{25a}$$

$$y^* = [\underline{u}^*,\sigma^*] \in E(\underline{\Psi}^*,\rho,C_p,C_s;\ \Omega \times T^+) \tag{25b}$$

defined on Ω with initial conditions:

$$\underline{u}(\underline{x},0) = \underline{u}_o(\underline{x}) \qquad \dot{\underline{u}}(\underline{x},0) = \underline{v}_o(\underline{x}) \tag{26a}$$

$$\underline{u}^*(\underline{x},0) = \underline{u}^*_o(\underline{x}) \qquad \dot{\underline{u}}^*(\underline{x},0) = \underline{v}^*_o(\underline{x}) \qquad \text{in } \Omega \tag{26b}$$

The elastodynamic equilibrium equations for each state are given as:

$$\sigma_{ij,j} + \beta_i = 0 \qquad \text{in } \Omega \tag{27a}$$

$$\sigma^*_{ij,j} + \beta^*_i = 0 \tag{27b}$$

where

$$\beta_i = \rho\left(\Psi_i - \frac{\partial^2 u_i}{\partial \tau^2}\right) \tag{28a}$$

$$\beta^*_i = \rho\left(\Psi^*_i - \frac{\partial^2 u^*_i}{\partial \tau^2}\right) \tag{28b}$$

Betti's reciprocal theorem for two distinct equilibrium states is given as (Sokolnikoff, 1956, Section 109):

$$\int_\Gamma t^*_i u_i \, dA + \int_\Omega \beta^*_i u_i \, dV = \int_\Gamma t_i u^*_i \, dA + \int_\Omega \beta_i u^*_i \, dV \tag{29}$$

Note that we can substitute Equation 28 into Equation 29. Since this expression is true for all $\tau \in T^+$, we can integrate from 0 to t to obtain:

$$\int_0^t \int_\Gamma t^*_i u_i \, dA \, d\tau + \int_0^t \int_\Omega \rho \Psi^*_i u_i \, dV \, d\tau - \int_0^t \int_\Omega \rho \frac{\partial^2 u^*_i}{\partial \tau^2} u_i \, dV \, d\tau =$$

$$\int_0^t \int_\Gamma t_i u^*_i \, dA \, d\tau + \int_0^t \int_\Omega \rho \Psi_i u^*_i \, dV \, d\tau - \int_0^t \int_\Omega \rho \frac{\partial^2 u_i}{\partial \tau^2} u^*_i \, dV \, d\tau \tag{30}$$

Integration by parts of the time derivative terms gives:

$$\int_0^t \frac{\partial^2 u^*_i(\underline{x},\tau)}{\partial \tau^2} u_i(\underline{x},\tau) \, d\tau = \dot{u}^*_i(\underline{x},t) \, u_i(\underline{x},t) - v^*_{oi}(\underline{x}) \, u_{oi}(\underline{x})$$

$$- \int_0^t \dot{u}^*_i(\underline{x},\tau) \, \dot{u}_i(\underline{x},\tau) \, d\tau \tag{31a}$$

$$\int_0^t \frac{\partial^2 u_i(\underline{x},\tau)}{\partial \tau^2}\, u_i^*(\underline{x},\tau)\, d\tau \;=\; \dot{u}_i(\underline{x},t)\, u_i^*(\underline{x},t) - v_{oi}(\underline{x})\, u_{oi}^*(\underline{x})$$

$$- \int_0^t \dot{u}_i(\underline{x},\tau)\, \dot{u}_i^*(\underline{x},\tau)\, d\tau \qquad (31b)$$

where $\dot{u}_i = du_i/d\tau$. Substituting these results into Equation 30 and showing the independent variables explicitly, the following form of the reciprocal theorem of elastodynamics is obtained:

$$\int_0^t \int_\Gamma t_i^*(\underline{x},\tau)\, u_i(\underline{x},\tau)\, dA(\underline{x})\, d\tau + \int_0^t \int_\Omega \rho(\underline{x})\, \Psi_i^*(\underline{x},\tau)\, u_i(\underline{x},\tau)\, dV(\underline{x})\, d\tau$$

$$- \int_\Omega \rho(\underline{x}) \left[\dot{u}_i^*(\underline{x},t)\, u_i(\underline{x},t) - v_{oi}^*(\underline{x})\, u_{oi}(\underline{x}) \right] dV(\underline{x})$$

$$= \int_0^t \int_\Gamma t_i(\underline{x},\tau)\, u_i^*(\underline{x},\tau)\, dA(\underline{x})\, d\tau + \int_0^t \int_\Omega \rho(\underline{x})\, \Psi_i(\underline{x},\tau)\, u_i^*(\underline{x},\tau)\, dV(\underline{x})\, d\tau$$

$$- \int_\Omega \rho(\underline{x}) \left[\dot{u}_i(\underline{x},t)\, u_i^*(\underline{x},t) - v_{oi}(\underline{x})\, u_{oi}^*(\underline{x}) \right] dV(\underline{x}) \qquad (32)$$

Mansur and Brebbia (1985) derive the elastodynamic equivalents of Somigliana's identities from this relationship. We will show the more classical approach obtaining the same integral equations via Graffi's theorem.

In obtaining Equation 32, we tacitly assumed that the two elastodynamic states occurred at the same time, τ. Alternatively, we can, in effect, integrate the y^* state from t down to 0; that is, the reciprocal theorem "compares" the y^* state at time $t' = t-\tau$ with the y state at time τ. Then the integration by parts of Equation 31 becomes:

$$\int_0^t \frac{\partial^2 u_i^*(\underline{x},t-\tau)}{\partial \tau^2}\, u_i(\underline{x},\tau)\, d\tau = u_i^*(\underline{x},o)\, u_i(\underline{x},t) - \dot{u}_i^*(\underline{x},t)\, u_{oi}(\underline{x}) - \int_0^t \dot{u}_i^*(\underline{x},t-\tau)\, \dot{u}_i(\underline{x},\tau)\, d\tau \qquad (33a)$$

$$\int_0^t \frac{\partial^2 u_i(\underline{x},\tau)}{\partial \tau^2}\, u_i^*(\underline{x},t-\tau)\, d\tau = \dot{u}_i(\underline{x},t)\, u_{oi}^*(\underline{x}) - v_{oi}(\underline{x})\, u_i^*(\underline{x},t) - \int_0^t \dot{u}_i(\underline{x},\tau)\, \dot{u}_i^*(\underline{x},t-\tau)\, d\tau \qquad (33b)$$

We want the time derivatives that are outside of the time integrations to be taken with respect to t instead of τ. Noting that,

$$\frac{\partial u_i^*(\underline{x},t-\tau)}{\partial \tau} = -\frac{\partial u_i^*(\underline{x},t-\tau)}{\partial t}$$

We now define:

$$\dot{u}_i = \partial u_i / \partial t$$

Combining the results of Equation 33 with Equation 30 and rearranging terms gives:

$$\int_\Gamma \int_0^t t_i^*(\underline{x},t-\tau)\, u_i(\underline{x},\tau)\, d\tau\, dA(\underline{x}) + \int_\Omega \rho(\underline{x}) \left[\int_0^t \Psi_i^*(\underline{x},t-\tau)\, u_i(\underline{x},\tau)\, d\tau + v_{oi}^*(\underline{x})\, u_i(\underline{x},t) + u_{oi}^*(\underline{x})\, \dot{u}_i(\underline{x},t) \right] dV(\underline{x})$$

$$= \int_{\Gamma} \int_{0}^{t} t_i(\underline{x},\tau)\, u_i^*(\underline{x},t-\tau)\, d\tau\, dA(\underline{x}) + \int_{\Omega} \rho(\underline{x}) \left[\int_{0}^{t} \Psi_i(\underline{x},\tau)\, u_i^*(\underline{x},t-\tau)\, d\tau \right.$$

$$\left. + v_{oi}(\underline{x})\, u_i^*(\underline{x},t) + u_{oi}(\underline{x})\, \dot{u}_i^*(\underline{x},t) \right] dV(\underline{x}) \qquad (34)$$

This is Graffi's reciprocal theorem of elastodynamics. Thus the dynamical reciprocal theorem is an extension of Betti's reciprocal theorem of elastostatics. Most authors express Equation 34 in terms of Riemann convolutions (see Nomenclature) as:

$$\int_{\Gamma} [t_i^* * u_i](\underline{x},t)\, dA(\underline{x}) + \int_{\Omega} \rho(\underline{x}) \left[[\Psi_i^* * u_i](\underline{x},t) \right.$$

$$\left. + v_{oi}^*(\underline{x})\, u_i(\underline{x},t) + u_{oi}^*(\underline{x})\, \dot{u}_i(\underline{x},t) \right] dV(\underline{x})$$

$$= \int_{\Gamma} [t_i * u_i^*](\underline{x},t)\, dA(\underline{x}) + \int_{\Omega} \rho(\underline{x}) \left[[\Psi_i * u_i^*](\underline{x},t) \right.$$

$$\left. + v_{oi}(\underline{x})\, u_i^*(\underline{x},t) + u_{oi}(\underline{x})\, \dot{u}_i^*(\underline{x},t) \right] dV(\underline{x}) \qquad (35)$$

In elastostatics, an integral equation statement of the boundary value problem (Somigliana's identities) is derived from Betti's reciprocal theorem by using as one of the equilibrium states a state given by the fundamental solution (Kelvin's solution). In the following section an integral equation statement of the elastodynamics problem is derived from Graffi's reciprocal theorem using Stokes's solution to define one of the elastodynamic states.

Integral Equations for the Direct Boundary Element Method. The integral equation statement derived in this section is an exact statement of the elastodynamics problem. The numerical approximation of this statement is presented in a later section.

We apply Graffi's theorem to two elastodynamic states: y corresponds to our elastodynamics problem and y^* corresponds to the Stokes state of quiescent past where the body load is a concentrated impulse load applied at $\tau=0$ (i.e., the Green's function results of Equations 21 through 24). Graffi's theorem is then written as:

$$\int_\Gamma \int_0^t F_{ik}(\underline{x},t-\tau;\underline{\xi})\, e_k(\underline{\xi})\, u_i(\underline{x},\tau)\, d\tau\, dA(x)$$

$$+ \int_\Omega \int_0^t \delta(t-\tau)\, \delta(\underline{x}-\underline{\xi})\, e_i(\underline{\xi})\, u_i(\underline{x},\tau)\, d\tau\, dV(\underline{x})$$

$$= \int_\Gamma \int_0^t t_i(\underline{x},\tau)\, G_{ik}(\underline{x},t-\tau;\underline{\xi})\, e_k(\underline{\xi})\, d\tau\, dA(\underline{x})$$

$$+ \rho \int_\Omega \left[\int_0^t \Psi_i(\underline{x},\tau)\, G_{ik}(\underline{x},t-\tau;\underline{\xi})\, e_k(\underline{\xi})\, d\tau \right.$$

$$\left. + v_{oi}(\underline{x})\, G_{ik}(\underline{x},t;\underline{\xi})\, e_k(\underline{\xi}) + u_{oi}(x)\, \dot{G}_{ik}(\underline{x},t;\underline{\xi})\, e_k(\underline{\xi}) \right] dV(\underline{x}) \qquad (36)$$

As indicated in Equations 22 and 23, the times t and τ always occur in the Green's function (and the corresponding higher order tensors) as the difference $t-\tau$. Physically, the response of the domain to a unit impulse is a function of the elaspsed time since the impulse has occurred. Mathematically, the Stokes solution (including the special case for the Green's function) has the property of time translation. In particular, let B_{ik} represent F_{ik} or G_{ik}, then we have:

$$B_{ik}(\underline{x},t-\tau;\underline{\xi}) = B_{ik}(\underline{x},t-\tau;\underline{\xi},0) = B_{ik}(\underline{x},t;\underline{\xi},\tau) \qquad (37)$$

By Equation 37 and the definition of the Dirac delta "function," Equation 36 becomes:

$$u_k(\underline{\xi},t) = \int_\Gamma \int_0^t \left[G_{ik}(\underline{x},t;\underline{\xi},\tau)\ t_i(\underline{x},\tau) - F_{ik}(\underline{x},t;\underline{\xi},\tau)\ u_i(\underline{x},\tau) \right] d\tau\ dA(\underline{x})$$

$$+ \int_\Omega \int_0^t G_{ik}(\underline{x},t;\underline{\xi},\tau)\ \psi_i(\underline{x},\tau)\ d\tau\ dV(\underline{x}) + I_k^U(\underline{\xi},t;\underline{v}_o,\underline{u}_o) \qquad (38a)$$

where

$$I_k^U(\underline{\xi},t;\underline{v}_o,\underline{u}_o) = \rho \int_\Omega \left[v_{oi}(\underline{x})\ G_{ik}(x,t,\xi) + u_{oi}(\underline{x})\ \dot{G}_{ik}(\underline{x},t;\underline{\xi}) \right] dV(\underline{x})$$

Note that $I_k^U(\underline{\xi},t;\underline{v}_o,\underline{u}_o)$ accounts for the effect of the initial conditions on the displacement field. Two alternative forms which differ only in notation are given below. The first form uses the convolution notation and is often written in the literature in an even more concise manner.

$$u_k(\underline{\xi},t) = \int_\Gamma \left\{ [G_{ik} * t_i](\underline{x},t) - [F_{ik} * u_i](\underline{x},t) \right\} dA(\underline{x})$$

$$+ \int_\Omega [G_{ik} * \psi_i](\underline{x},t)\ dV(\underline{x}) + I_k^U(\underline{\xi},t;\underline{v}_o,\underline{u}_o) \qquad (38b)$$

The second form is expressed in terms of functional operators for $y_k^o(\underline{x},t;\underline{\xi}|f)$, the Stokes state of quiescent past (Eringin and Suhubi, 1975, Section 5.11). It is given as:

$$u_k(\underline{\xi},t) = \int_\Gamma \left\{ u_{ik}^o[\underline{x},t;\underline{\xi}|t_i(\underline{x},t)] - t_{ik}^o[x,t;\xi|u_i(\underline{x},t)] \right\} dA(\underline{x})$$

$$+ \int_\Omega u_{ik}^o[\underline{x},t;\underline{\xi}|\psi_i(\underline{x},t)]\ dV(x) + I_k^U(\underline{\xi},t;\underline{v}_o,\underline{u}_o) \qquad (38c)$$

where, by Cauchy's formula:

$$t_{ik}^o[\underline{x},t;\underline{\xi}|f] = \sigma_{ijk}^o[\underline{x},t;\underline{\xi}|f]\ n_j(x)$$

The equivalence of the functional operator form (Equation 38c) to the time convolution form (Equation 38a or 38b) is illustrated by Eringen and Suhubi (1975). Equation 38 is sometimes referred to as Love's (1944) integral identity (Eringen and Suhubi, 1975, give additional historical background). Love (1944) considered $I_k^U(\xi,t;\underline{v}_o,\underline{u}_o)$ in more detail showing that the initial displacement and velocity at a point $\underline{x}$ affects a region bounded by two spheres traveling at wave velocities C_P and C_S centered at $\underline{x}$ (see Eringen and Suhubi, 1975). Equation 38 is the elastodynamic counterpart to Somigliana's first identity of elastostatics and plays the same role in development of the direct BEM for elastodynamics.

The equation does not represent a solution to the elastodynamic problem since the traction and displacement distribution along the boundary are only partially known. (For a well posed problem, only "half" of the boundary information is known.) However, when applied to the boundary, it provides an alternative analytical statement of the boundary-initial value problem (BIVP). Consider applying Equation 38 to an arbitrary point ξ_Γ on Γ:

$$u_k(\xi_\Gamma,t) = \int_\Gamma [G_{ik}*t_i](\underline{x},t)\, dA(\underline{x}) - \int_\Gamma [F_{ik}*u_i](\underline{x},t)\, dA(\underline{x}) + \int_\Omega [G_{ik}*\psi_i](\underline{x},t)\, dV(x) + I_k^U(\xi_\Gamma,t;\underline{v}_c,\underline{u}_o) \qquad (39)$$

Note that the second boundary integration involves an improper integral to be interpreted as a Cauchy principal value intergral. In general this can be written as:

$$\int_\Gamma [F_{ik}*u_i](x,t)\, dA(x) = F_{ik}^{SING}\, u_i(\xi_\Gamma,t) + \int_\Gamma [F_{ik}*u_i](\underline{x},t)\, dA(\underline{x}) \qquad (40)$$

where F_{ik}^{SING} is the singularity contribution and the last integral must be intepreted as a Cauchy principal value integral. For smooth boundaries $F_{ik}^{SING} = -1/2\ \delta_{ik}$ (see e.g., Cole, Kosloff, and Minster, 1978, Appendix A) and Equation 39 becomes:

$$\frac{1}{2}\, u_k(\xi_\Gamma,t) = \int_\Gamma \left\{ [G_{ik}*t_i](x,t) - [F_{ik}*u_i](\underline{x},t) \right\} dA(\underline{x})$$

$$+ \int_\Omega [G_{ik}*\psi_i](x,t)\, dV(x) + I^U_k(\xi_\Gamma,t;\underline{v}_o,\underline{u}_o) \qquad (41)$$

This gives us a boundary integral equation (BIE) statement of the boundary-initial value problem. Relations similar to Love's integral identity can be obtained for strain, stress, and traction by applying Equations 2 and 3, and Cauchy's formula, to Equation 38; in a numerical setting these suggested relations are often not used. The numerical approximation of the above integral equations will be considered in subsequent sections. First, we consider the integral equations which the indirect boundary element method (IBEM) approximates in the next section.

Integral Equations for the Indirect Boundary Element Method. The integral equations which the IBEM numerically approximates can be deduced from Equation 38. Banerjee and Butterfield (1981) present the analogous development for steady-state and transient potential flow using an idea originally due to Lamb (1932). Eringen and Suhubi (1975, Section 5.14) derive the integral equations for elastodynamics by the same argument. An overview of the derivation is presented below.

The indirect formulation can physically be visualized as embedding the domain of the problem in an infinite space of the same media. The traction distribution, along the surface corresponding to the boundary of the original problem, is then sought which will satisfy the boundary conditions.

The integral equations for this method can be derived from Equation 38 by considering two displacement boundary-initial value problems:

1. A displacement BIVP with body forces and initial conditions defined on a domain Ω and specified boundary conditions on the boundary Γ.

2. A displacement BIVP in the compliment space Ω^C having the same boundary conditions but with nil body forces and initial conditions.

For a finite domain problem with boundary Γ, Ω and Ω^C correspond to the domains for the interior and exterior problems, respectively. Applying Equation 38 to both problems and equating the two expressions (knowing the boundary conditions are equal) gives the relation:

$$u_i(\underline{x},t) = \int_\Gamma [G_{ik}*P_k](\underline{\xi},t)\, dA(\underline{\xi}) + \int_\Omega [G_{ik}*\psi_k](\underline{\xi},t)\, dV(\underline{\xi}) + I_i^U(\underline{x},t;\underline{v}_o,\underline{u}_o) \qquad (42a)$$

where the unknown vector valued function $P_k(\underline{\xi},\tau)$ is the difference in the boundary traction distributions for the two problems, given as:

$$P_k(\xi,\tau) = t_k(\underline{\xi},\tau) - t_k^C(\underline{\xi},\tau) \qquad (\xi,\tau) \in \Gamma \times T^+ \qquad (42b)$$

$P_k(\underline{\xi},\tau)$ is often referred to as the fictitious or artificial traction distribution, it's artificial in the sense that it is a consequence of embedding the problem in the infinite domain and has no meaning outside of this context. To obtain the above relationship, symmetry of the Green's function with respect to its indices and spacial arguments was employed.

Given the expression for the displacement field in Equation 42a, the kinematic equation (Equation 2), constitutive relation (Equation 3), and Cauchy's formula then give the traction field for a unit normal $n_j(\underline{x})$ as:

$$t_i(\underline{x},t) = \int_\Gamma [F_{ik}*P_k](\underline{\xi},t)\, dA(\underline{\xi}) + \int_\Omega [F_{ik}*\psi_k](\underline{\xi},t)\, dV(\underline{\xi}) + I_k^T(\underline{x},t;\underline{v}_o,\underline{u}_o) \qquad (43)$$

where

$$I_i^T(\underline{x},t;\underline{v}_o,\underline{u}_o) = \rho \int_\Omega v_{ok}(\underline{\xi})\, F_{ik}(x,t;\xi) + u_{ok}(\underline{\xi})\, \dot{F}_{ik}(\underline{x},t;\underline{\xi})]\, dV(\underline{\xi})$$

Similar expressions for the stress and strain fields are intermediate steps in obtaining Equation 43, but often these expressions are not used in the numerical formulation. If the artificial traction distribution were known, the response in the domain would be completely defined by integral equations (e.g., Equation 42 gives the displacement in the domain). To attain the artificial tractions, we bring the response point to the boundary and enforce the boundary conditions.

To obtain the key integral equation which the IBEM is based on, we assumed a displacement BIVP. In actuality, we apply the IBEM to the fully-mixed BIVP and, thus, must consider traction boundary conditions also. Note that while Equation 42 is regular upon integration, Equation 43 must be interpreted in a Cauchy principal value sense. The boundary integral equations are then given as:

$$\hat{u}_i(\underline{x}_\Gamma,t) = \int_\Gamma [G_{ik}*\bar{P}_k](\underline{\xi},t)\ dA(\underline{\xi}) + \int_\Omega [G_{ik}*\hat{\psi}_k](\underline{\xi},t)\ dV(\underline{\xi}) + I_i^U(\underline{x}_\Gamma,t;\hat{\underline{v}}_o,\hat{\underline{u}}_o) \qquad \underline{x}_\Gamma \in \Gamma_U \tag{44a}$$

$$\hat{t}_i(\underline{x}_\Gamma,t) = \pm \frac{1}{2}\delta_{ik}\bar{P}_k(\underline{x}_\Gamma,t) + \int_\Gamma [F_{ik}*\bar{P}_k](\underline{\xi},t)\ dA(\underline{\xi}) + \int_\Omega [F_{ik}*\hat{\psi}_k](\underline{\xi},t)\ dV(\underline{\xi}) + I_i^T(\underline{x}_\Gamma,t;\hat{\underline{v}}_o,\hat{\underline{u}}_o) \qquad \underline{x}_\Gamma \in \Gamma_T \tag{44b}$$

where "^" and "-" have been used to explicitly indicate the known and unknown field variables, respectively. In Equation 44b, the singular contribution is based upon a smooth boundary and the sign depends upon the orientation of the normal vector.

This integral equation approach is sometimes referred to as "an integral equation representation by vector simple-layer potentials," which reflects its potential theory origins (see e.g., Jaswon and Symm, 1977). Equations 44a and 44b are vector Fredholm integral equations of the first and second kinds, respectively.

Among the differences between the integral equations on which the DBEM and IBEM are based, note that for the IBEM integration of boundary displacements has been eliminated and thus only a single boundary integration in terms of the artificial tractions remains. Because of the symmetry in the Green's tensor, this provides a field-source interpretation to the integral equations -- a physically meaningful interpretation.

In the following sections we consider the numerical approximation of the integral equations developed above and how these numerical approximations might be combined with those of the FEM.

NUMERICAL SOLUTION

The above integral equation formulations provide a potentially effective approach to modeling infinite domains. In the first subsections below, we will consider the numerical approximations made in the integral equations to obtain the direct and indirect BEMs. The third subsection addresses how the efficiency of both BEMs can be "optimized" by exploiting properties of the free-space Green's function and modern computer hardware. The final subsection gives an overview of coupling BEMs with the FEM by treating the BEM as a nonlinear boundary condition on the FEM subregion of the problem.

Approximation of Integral Equations -- Boundary Element Methods

The following approximations are common to both BEMs: (1) the integrations are performed in a piecewise manner, and (2) the boundary integral equations are approximately satisfied in a boundary weighted residual sense (usually by collocation). Though by tradition we appear to be stuck with the name "boundary element methods," it is somewhat of a misnomer; it suggests that analagous to finite elements (on the domain) we will have boundary elements with known shape functions which constrain the boundary displacement. We often use the same locally supported family of polynomials in the BEM as the FEM uses as shape functions (Lachat and Watson, 1975); however, these functions merely facilitate the piecewise

integrations and do not imply that the fields are actually constrained to these interpolation functions. This implies that a coupling between the BE and FE methods will be inherently incompatible. For a numerical demonstration of this incompatibility in elastostatics see Cox (1988).

Let's consider the interpolation of boundary and domain quantities before addressing a specific BEM. (The presentation given here is intended for individuals familiar with the discretizations commonly made in BEMs; for more detail, see Banerjee and Butterfield, 1981 or Brebbia, Telles, and Wrobel, 1984.) The approximations for the displacement and traction in terms of expansion functions in time and space (on Γ) are given as:

$$u_i(\underline{x}_\Gamma,t) = \sum_{n=1}^{N_{NP}} \sum_{s=1}^{N_{TS}} N_n(\underline{x}_\Gamma)\, T_s(t)\, \bar{u}_i^{ns} \qquad \underline{x}_\Gamma \in \Gamma \tag{45a}$$

$$t_i(\underline{x}_\Gamma,t) = \sum_{n=1}^{N_{NP}} \sum_{s=1}^{N_{TS}} N_n(\underline{x}_\Gamma)\, T_s(t)\, \bar{t}_i^{ns} \tag{45b}$$

Similarly if we let $\chi_i(\underline{x}_\Omega,t)$ represent any of the vector-valued field quantities in Equations 38 through 44, it can be approximated as:

$$\chi_i(\underline{x}_\Omega,t) = \sum_{m=1}^{N_{MP}} \sum_{s=1}^{N_{TS}} M_m(\underline{x}_\Omega)\, T_s(t)\, \bar{\chi}_i^{ms} \qquad \underline{x}_\Omega \in \Omega \tag{45c}$$

where $N_n(\underline{x}_\Gamma)$ = n^{th} expansion function for traction or displacement on Γ

$M_m(\underline{x}_\Omega)$ = m^{th} expansion function for a field variable on Ω

$T_s(t)$ = s^{th} expansion function for time

N_{NP} = number of boundary expansion functions

N_{MP} = number of domain expansion functions

N_{TS} = number of time expansion functions

$\bar{u}_i^{ns}$, $\bar{t}_i^{ns}$, $\bar{\chi}_i^{ms}$ = expansion coefficients for the corresponding quantities

In the more general case, each boundary and domain expansion could be written in terms of unique expansion functions. For implementation purposes, let's be more specific instead of more general. The time domain of the analysis $[0,T^A]$ is subdivided into N_{TS} time intervals where the time at the end of the j^{th} interval is denoted as t_j. In general, it is not necessary that these time intervals (or steps) be of uniform duration; however, as we will see in a later section, uniform steps allow a more efficient time-stepping procedure. The boundary is discretized into N_{EL} surfaces which (despite the misnomer) will be referred to as elements; the union of these surfaces spans the boundary. Associated with the elements are N_{NP} nodal points occurring at the extremeties or within the elements. Similarly, the domain is discretized into N_{CEL} volume cells; the union of which spans the portions of the domain where nonzero body forces or initial conditions occur. Associated with the cells are N_{MP} cell points occurring on the verticies, edges, and within the cells. Geometrically, the boundary elements and volume cells are similar to FE shell and solid (e.g., brick) elements. So that the expansion coefficients will correspond to the nodal values of the corresponding boundary values, the following is required:

$$N_n(\underline{x}_j) = \delta_{nj} \qquad \underline{x}_j \in \Gamma \tag{46a}$$

$$T_s(t_j) = \delta_{sj} \tag{46b}$$

where $\underline{x}_j$ = position vector for the j^{th} nodal point. Additionally, for the expansion coefficients of the domain to correspond to the values at the cell points, we require:

$$M_m(\underline{x}_j) = \delta_{mj} \qquad \underline{x}_j \in \Omega \tag{46c}$$

where $\underline{x}_j$ = position vector for the j^{th} cell point.

So, as an example, $\bar{u}_i^{ns}$ of Equation 45a is the displacement vector at node n and time step s. As suggested at the beginning of this section, the interpolation functions in space often correpond to the polynomial shape functions of the FEM. Before proceeding, let's discuss a few important exceptions which clarify the terminology:

1. An exception to this occurs with so-called "discontinuous elements." The simplest example is the constant element where the integration over the element treats the boundary value as a constant. In this case the nodal point resides at the center of the element. Note that the use of the term "node" here corresponds to a point where boundary values are specified and, thus, for the constant element is not sufficient to also define the geometry. With "discontinuous elements" it is easy to define loading discontinuities if they occur along element boundaries; adjoining elements simply have distinct node points approaching the element boundary. In practice we let these points coincide and simply perform the necessary bookkeeping to associate the corresponding boundary values with the correct element. "Discontinuous elements" will be further discussed in the next section with regard to the collocation technique.

2. Another exception occurs for infinite domains where an interface between BE regions extends to infinity, special interpolation functions are required to formulate so-called "infinite boundary elements."

All of these interpolation functions have the mathematical attribute referred to as "local support;" though they are defined over the whole domain/boundary, they are nonzero only in adjoining cells/elements. Local support will not provide a sparse system of equations as with the FEM, however it does reduce assembly effort in forming the systems of algebraic equations that approximate the integral equations.

The expansion functions in time can also be thought of as having local support. These functions are defined such that:

$$T_s(t) = \begin{cases} 1 , & t = t_s \\ \sum_{i=0}^{n} a_i \, t^i , & t \in (t_{s-1}, t_s) \\ 0 , & t \notin (t_{s-1}, t_s] \end{cases}$$

That is, they are polynomial functions on the time step preceeding t_s. Different choices of the polynomial coefficients give various integration rules in time.

These interpolations are the first of two (principal) approximations to be made. The second approximation addresses how the unknown boundary values are obtained; how do we obtain the "best" expansion coefficients?

Direct Boundary Element Method. This section addresses the numerical approximation of integral equations 38 and 41 -- the direct boundary element method. As a stand-alone analysis tool or in a coupled solution, Equation 41 must be approximated on the boundary to attain the unknown boundary values. In this section we consider the application of the DBEM as a stand-alone analysis tool; many of the equations are common to the coupled solution approach. The response within the domain at any point can then be calculated by integral equations like Equation 38. Using the interpolations of the previous section, the boundary integral equation (Equation 41) is approximated as:

$$\frac{1}{2}\, u_k(\underline{\xi},t) = \sum_{n=1}^{N_{NP}} \sum_{s=1}^{j} \left(g_{ik}^{ns}(\underline{\xi},t)\; t_i^{ns} - f_{ik}^{ns}(\underline{\xi},t)\; u_i^{ns} \right)$$

$$+ \sum_{m=1}^{N_{MP}} \sum_{s=1}^{j} {}' g_{ik}^{ms}(\underline{\xi},t)\; \hat{\psi}_i^{ms} + \sum_{m=1}^{N_{MP}} \left(\tilde{\underline{g}}_{ik}^{m}(\underline{\xi},t)\; \hat{v}_{oi}^{m} \right.$$

$$\left. + \tilde{\tilde{g}}_{ik}^{m}(\underline{\xi},t)\; \hat{u}_{oi}^{m} \right) \qquad \xi \varepsilon \Gamma \;\Lambda\; t \varepsilon (t_{j-1}, t_j] \qquad (47a)$$

where

$$g_{ik}^{ns}(\underline{\xi},t) = \int_{\Gamma} \int_{0}^{t} G_{ik}(\underline{x},t;\underline{\xi},\tau)\; N_n(x)\; T_s(\tau)\; d\tau\; dA(\underline{x}) \qquad (47b)$$

$$f_{ik}^{ns}(\underline{\xi},t) = \int_{\Gamma}\int_{0}^{t} F_{ik}(\underline{x},t;\underline{\xi},\tau)\, N_n(\underline{x})\, T_s(\tau)\, d\tau\, dA(\underline{x}) \qquad (47c)$$

$$'g_{ik}^{ms}(\underline{\xi},t) = \int_{\Omega}\int_{0}^{t} G_{ik}(\underline{x},t;\underline{\xi},\tau)\, M_m(\underline{x})\, T_s(\tau)\, d\tau\, dV(\underline{x}) \qquad (47d)$$

$$\tilde{g}_{ik}^{m}(\underline{\xi},t) = \int_{\Omega} G_{ik}(\underline{x},t;\underline{\xi})\, M_m(\underline{x})\, dV(\underline{x}) \qquad (47e)$$

$$\tilde{\tilde{g}}_{ik}^{m}(\underline{\xi},t) = \int_{\Omega} \dot{G}_{ik}(\underline{x},t;\underline{\xi})\, M_m(\underline{x})\, dV(\underline{x}) \qquad (47f)$$

The local support of each interpolation function tacitly implies reduced limits of integration in both time and space.

The nodal boundary values at the time t_j are approximated by satisfying the integral equation in a collocation sense with respect to both time and space. Other boundary weighted residual techniques could be used but collocation is the most prevalent. In time, collocation occurs at the ends of the previously defined time steps; in space, the nodes are the most common collocation points. Some researchers (see e.g., Patterson and Sheikh, 1981 or Brebbia, Telles and Wrobel, 1984) only collocate inside the elements to: (1) eliminate special singular contribution calculations when geometric discontinuities coincide with nodes, and (2) to simplify the element assembly procedures. Internally collocated elements are often referred to as "discontinuous" or "non-conforming" elements -- another misnomer based on the shape function fallacy. The following development does not exclude nodal collocation; however, the singular contribution used to obtain Equation 41 would change for a boundary point which does not have a unique tangent plane.

Applying Equation 47 to N_{CP} collocation points at the end of the j^{th} time step gives:

$$\frac{1}{2}\,\underline{u}_c^j = \sum_{n=1}^{N_{NP}} \sum_{s=1}^{j} \left(DG_{cn}^{js}\, \underline{t}_n^s - {}^{*}DF_{cn}^{js}\, \underline{u}_n^s \right) + \underline{\hat{R}}_c^j \qquad c = 1,2\ldots,N_{CP} \qquad (48a)$$

where

$$\mathbf{DG}_{cn}^{js} = \left[g_{ik}^{ns}(\xi_c, t_j) \right]^T \varepsilon \; \mathbf{R}^{3x3} \tag{48b}$$

$$^{*}\mathbf{DF}_{cn}^{js} = \left[f_{ik}^{ns}(\xi_c, t_j) \right]^T \varepsilon \; \mathbf{R}^{3x3} \tag{48c}$$

$$\underline{t}_n^s = \left\{ t_i^{ns} \right\} \varepsilon \; \mathbf{R}^3 \tag{48d}$$

$$\underline{u}_n^s = \left\{ u_i^{ns} \right\} \varepsilon \; \mathbf{R}^3 \tag{48e}$$

$$\hat{\underline{R}}_c^j = \left\{ \sum_{m=1}^{N_{MP}} \sum_{s=1}^{j} {}' g_{ik}^{ms}(\xi_c, t_j) \; \hat{\psi}_i^{ms} + \sum_{m=1}^{N_{MP}} \left(\tilde{g}_{ik}^{m}(\xi_c, t_j) \; \hat{v}_{oi}^{m} + \tilde{\tilde{g}}_{ik}^{m}(\xi_c, t_j) \; \hat{u}_{oi}^{m} \right) \right\} \varepsilon \; \mathbf{R}^3 \tag{48f}$$

Note that the vector $\underline{R}_c^j$ gives the effect of body forces and initial conditions on the displacement at the collocation point. By combining the singular contribution term on the left-hand side with the appropriate $^{*}\mathbf{DF}$ matrices, one obtains:

$$\underline{0} = \sum_{n=1}^{N_{NP}} \sum_{s=1}^{j} \left(\mathbf{DG}_{cn}^{js} \; \underline{t}_n^s - \mathbf{DF}_{cn}^{js} \; \underline{u}_n^s \right) + \hat{\underline{R}}_c^j \qquad c = 1,2,\ldots,N_{CP} \tag{49}$$

If the collocation point coincides with the p^{th} node, the relationship between $^{*}\mathbf{DF}$ and $\mathbf{DF}$ is given as:

$$\mathbf{DF}_{cn}^{js} = {}^{*}\mathbf{DF}_{cn}^{js} + \frac{1}{2} \; \mathbf{I} \; \delta_{sj} \; \delta_{np}$$

where I is the identity matrix for $\mathbf{R}^3$. Combining the equations associated with each collocation point into a single system of equations gives:

$$\underline{0} = \sum_{s=1}^{j-1} \left(\mathbf{G}^s \, \hat{\underline{t}}^s - \mathbf{F}^s \, \hat{\underline{u}}^s \right) + \mathbf{G}^j \, \underline{t}^j - \mathbf{F}^j \, \underline{u}^j + \hat{\underline{R}}^j \tag{50}$$

where

$$\mathbf{G}^s = \begin{bmatrix} \mathbf{DG}_{11}^{js} & \mathbf{DG}_{12}^{js} & \cdots & \mathbf{DG}_{1N_{NP}}^{js} \\ \mathbf{DG}_{21}^{js} & & & \vdots \\ \vdots & & & \vdots \\ \mathbf{DG}_{N_{CP}1}^{js} & \cdots & & \mathbf{DG}_{N_{CP}N_{NP}}^{js} \end{bmatrix} \qquad s = 1,2,\ldots,j$$

$$\mathbf{F}^s = \begin{bmatrix} \mathbf{DF}_{11}^{js} & \mathbf{DF}_{12}^{js} & \cdots & \mathbf{DF}_{1N_{NP}}^{js} \\ \mathbf{DF}_{21}^{js} & & & \vdots \\ \vdots & & & \vdots \\ \mathbf{DF}_{N_{CP}1}^{js} & \cdots & & \mathbf{DF}_{N_{CP}N_{NP}}^{js} \end{bmatrix} \qquad s = 1,2,\ldots,j$$

$$\underline{t}^s = \begin{Bmatrix} \underline{t}_1^s \\ \underline{t}_2^s \\ \vdots \\ \underline{t}_{N_{NP}}^s \end{Bmatrix}, \quad \underline{u}^s = \begin{Bmatrix} \underline{u}_1^s \\ \underline{u}_2^s \\ \vdots \\ \underline{u}_{N_{NP}}^s \end{Bmatrix}, \quad \hat{\underline{R}}^j = \begin{Bmatrix} \hat{\underline{R}}_1^j \\ \hat{\underline{R}}_2^j \\ \vdots \\ \hat{\underline{R}}_{N_{CP}}^j \end{Bmatrix} \quad s = 1,2,..,j$$

Assuming $N_{CP} = N_{NP}$, we have $\mathbf{G}^s$, $\mathbf{F}^s \in \mathbf{R}^{N_{EQ} \times N_{EQ}}$ and $\underline{t}^s$, $\underline{u}^s$, $\hat{\underline{R}}^j \in \mathbf{R}^{N_{EQ}}$ for all s where $N_{EQ} = 3N_{CP} = 3N_{NP}$. As an alternative one could over-collocate the integral equations ($N_{CP} > N_{NP}$, see e.g., Hutchinson, 1985)

and then solve the resulting overdetermined system in a least squares sense; however, its unlikely the overcollocation would be more effective than solving the integral equations in a Galerkin sense.

As explicitly denoted by the $\hat{\cdot}$'s, the only unknowns are half the boundary values at the current time step; actually this is an induction assumption which we know is valid for j=1. In the following equation, the induction is validated and a time stepping procedure is given. Partitioning the boundary value vectors for the j^{th} time step as:

$$\underline{0} = \sum_{s=1}^{j-1} \left(\mathbf{G}^s \, \hat{\underline{t}}^s - \mathbf{F}^s \, \hat{\underline{u}}^s \right) + \left[\mathbf{G}_1^j \; \mathbf{G}_2^j \right] \begin{Bmatrix} \bar{\underline{t}}^j \\ \hat{\underline{t}}^j \end{Bmatrix} - \left[\mathbf{F}_1^j \; \mathbf{F}_2^j \right] \begin{Bmatrix} \hat{\underline{u}}^j \\ \bar{\underline{u}}^j \end{Bmatrix} + \hat{\underline{R}}^j \qquad (51)$$

and isolating the unknowns on the left-hand side gives the following set of equations for the unknown boundary values:

$$\left[\mathbf{G}_1^j - \mathbf{F}_2^j \right] \begin{Bmatrix} \bar{\underline{t}}^j \\ \bar{\underline{u}}^j \end{Bmatrix} = \left[\mathbf{F}_1^j - \mathbf{G}_2^j \right] \begin{Bmatrix} \hat{\underline{u}}^j \\ \hat{\underline{t}}^j \end{Bmatrix} - \sum_{s=1}^{j-1} \left(\mathbf{G}^s \, \hat{\underline{t}}^s - \mathbf{F}^s \, \hat{\underline{u}}^s \right) - \hat{\underline{R}}^j \qquad (52)$$

The left-hand side now consists of the product of a fully-populated coefficient matrix with the vector of unknown boundary values at time t_j. The right-hand side of the system of equations is determined by four physically meaningfull sets of values: (1) the known boundary values at time t_j, (2) all the boundary values (displacements and tractions) at previous time steps, (3) the time variation of the body loads to time t_j, and (4) the initial conditions.

The unknown boundary values at time t_j are obtained by solving the above system of linear equations. With an approximation of the boundary values, the response at any point in the domain over the period of the j^{th} time step can be determined. Equation 38 gives the necessary integral equation for the displacement field. The numerical approximation of Equation 38 parallels the numerical approximation of Equation 41 except, instead of considering a boundary collocation point, we consider an internal (domain) response point. The following revisions of Equation 47 yield the numerical approximation of Equation 38: (1) eliminate the 1/2 factor on the left-hand side, and (2) interpret Equation 47c as a regular integration since $\underline{\xi} \in \Omega$.

As noted in a previous section, similar relations for the stress and strain fields can be easily derived. If one wanted both the stress and displacement fields in a region, it would probably be more efficient to: (1) calculate the displacement at a regular grid of points in the domain, and then (2) approximate the stresses by finite difference or finite element approximations. In the later case the derivatives operate on the interpolation functions which span regions between points of known displacement.

Equation 52 reflects the numerical burden that is characteristic of formulations which use integral equations in time -- convolution. The following section will discuss some ideas which significantly lighten this burden. These ideas on improving the numerical efficiency are equally applicable to the indirect BEM, so let's first consider its formulation in the following section.

Indirect Boundary Element Method. This section addresses the numerical approximation of integral Equations 42 through 44 -- the indirect boundary element method. As a stand-alone analysis tool or in a coupled solution, Equations 42 and 43 must be approximated on the boundary to attain the artificial boundary tractions. In this section we consider the application of the IBEM as a stand-alone analysis tool; many of the equations are common to the coupled solution approach. In a coupled solution approach, the equations "are applied a second time" to determine the unknown boundary values. The amount of numerical effort is essentially equal to the direct method; however, the indirect method calculates the artificial tractions as an intermediate step -- thus its name. With the artificial tractions, the response in the domain at any point can be calculated by integral equations like Equation 42a. Using the interpolations of Equation 45, boundary integral Equation 44 is written as:

$$\hat{v}_i(\underline{x},t) = \sum_{n=1}^{N_{NP}} \sum_{s=1}^{j} \left(g_{ik}^{ns}(\underline{x},t)\ P_k^{ns} \right) + \sum_{m=1}^{N_{MP}} \sum_{s=1}^{j} {}' g_{ik}^{ms}(\underline{x},t) \hat{\psi}_k^{ms}$$

$$+ \sum_{m=1}^{N_{MP}} \left(\tilde{g}_{ik}^{m}(\underline{x},t)\ \hat{v}_{ok}^{m} + \tilde{\tilde{g}}_{ik}^{m}(\underline{x},t)\ \hat{u}_{ok}^{m} \right)$$

$$\underline{x} \varepsilon \Gamma_U \wedge t \varepsilon (t_{j-1}, t_j] \tag{53a}$$

where

$$g_{ik}^{ns}(\underline{x},t) = \int_{\Gamma} \int_{0}^{t} G_{ik}(\underline{x},t;\underline{\xi},\tau)\ N_n(\underline{\xi})\ T_s(\tau)\ d\tau\ dA(\underline{\xi}) \tag{53b}$$

$${}' g_{ik}^{ms}(\underline{x},t) = \int_{\Omega} \int_{0}^{t} G_{ik}(\underline{x},t;\underline{\xi},\tau)\ M_m(\underline{\xi})\ T_s(\tau)\ d\tau\ dV(\underline{\xi}) \tag{53c}$$

$$\tilde{g}_{ik}^{m}(\underline{x},t) = \int_{\Omega} G_{ik}(\underline{x},t;\underline{\xi})\ M_m(\underline{\xi})\ dV(\underline{\xi}) \tag{53d}$$

$$\tilde{\tilde{g}}_{ik}^{m}(\underline{x},t) = \int_{\Omega} \dot{G}_{ik}(\underline{x},t;\underline{\xi})\ M_m(\underline{\xi})\ dV(\underline{\xi}) \tag{53e}$$

Since the free space Green's function is symmetric with respect to its spacial arguments and indices, the above integrations are equivalent to those of Equation 47. In a similar manner, Equation 44b is approximated as:

$$\hat{t}_i(\underline{x},t) = \pm \frac{1}{2} \delta_{ik}\ \bar{P}_k(\underline{x},t) + \sum_{n=1}^{N_{NP}} \sum_{s=1}^{j} \left(f_{ik}^{ns}(\underline{x},t)\ P_k^{ns} \right)$$

$$+ \sum_{m=1}^{N_{MP}} \sum_{s=1}^{j} {}' f_{ik}^{ms}(\underline{x},t)\ \hat{\psi}_k^{ms} + \sum_{m=1}^{N_{MP}} \left(\tilde{f}_{ik}^{m}(\underline{x},t)\ \hat{v}_{ok}^{m} + \tilde{\tilde{f}}_{ik}^{m}(\underline{x},t)\ \hat{u}_{ok}^{m} \right)$$

$$\underline{x} \varepsilon \Gamma_T \wedge t \varepsilon (t_{j-1}, t_j] \tag{54a}$$

where

$$f_{ik}^{ns}(\underline{x},t) = \int_{\Gamma}\int_{0}^{t} F_{ik}(\underline{x},t;\underline{\xi},\tau)\, N_n(\xi)\, T_s(\tau)\, d\tau\, dA(\underline{\xi}) \tag{54b}$$

$$'f_{ik}^{ms}(\underline{x},t) = \int_{\Omega}\int_{0} F_{ik}(\underline{x},t;\underline{\xi},\tau)\, M_m(\xi)\, T_s(\tau)\, d\tau\, dV(\underline{\xi}) \tag{54c}$$

$$\tilde{f}_{ik}^{m}(\underline{x},t) = \int_{\Omega} F_{ik}(\underline{x},t;\underline{\xi})\, M_m(\xi)\, dV(\xi) \tag{54d}$$

$$\tilde{\tilde{f}}_{ik}^{m}(\underline{x},t) = \int_{\Omega} \dot{F}_{ik}(\underline{x},t;\underline{\xi})\, M_m(\xi)\, dV(\xi) \tag{54e}$$

As previously mentioned, the local support of the interpolation functions tacitly implies reduced limits of integration in both time and space.

Again we will limit the discussion to the collocation satisfaction of the boundary integral equations. With the IBEM we apply Equation 53a to collocation points on Γ_U and apply Equation 54a to collocation points on Γ_T. As previously mentioned, we are actually addressing the fully-mixed BIVP; thus, the notation Γ_U and Γ_T is symbolic. Applying Equations 53a and 54a to N_{CP} collocation points at the end of the j^{th} time step gives:

$$\hat{\underline{u}}_c^j = \sum_{n=1}^{N_{NP}} \sum_{s=1}^{j} \mathbf{DG}_{cn}^{js}\, \underline{P}_n^s + \hat{\underline{RG}}_c^j \qquad c = 1,2,\ldots,N_{CP} \tag{55a}$$

where

N_U ~ number of displacement boundary conditions at collocation point c, $0 \le N_U \le 3$

$$\mathbf{DG}_{cn}^{js} = \left[g_{ik}^{ns}(\underline{x}_c,t_j)\right] \varepsilon\, \mathbf{R}^{N_U x 3} \tag{55b}$$

$$\hat{\underline{u}}_c^j = \left\{\hat{u}_i(\underline{x}_c,t_j)\right\} \varepsilon\, \mathbf{R}^{N_U} \tag{55c}$$

$$\underline{P}_n^s = \left\{P_k^{ns}\right\} \varepsilon \mathbf{R}^3 \tag{55d}$$

$$\underline{\hat{RG}}_c^j = \left\{ \sum_{m=1}^{N_{MP}} \sum_{s=1}^{j} {}' g_{ik}^{ms}(\underline{x}_c, t_j)\ \hat{\psi}_k^{ms} + \sum_{m=1}^{N_{MP}} \left(\tilde{g}_{ik}^{m}(\underline{x}_c, t_j)\ \hat{v}_{ok}^{m} + \tilde{\tilde{g}}_{ik}^{m}(\underline{x}_c, t_j)\ \hat{u}_{ok}^{m} \right) \right\} \varepsilon \mathbf{R}^{N_U} \tag{55e}$$

and

$$\underline{\hat{t}}_c^j = \pm \frac{1}{2}\ \underline{P}_c^j + \sum_{n=1}^{N_{NP}} \sum_{s=1}^{j} {}^{*}\mathbf{DF}_{cn}^{js}\ \underline{P}_n^s + \underline{\hat{RF}}_c^j \qquad c = 1,2,\ldots,N_{CP} \tag{56a}$$

where

N_T ~ number of traction boundary conditions at collocation point c, $0 \le N_T \le 3$

$${}^{*}\mathbf{DF}_{cn}^{js} = \left[f_{ik}^{ns}(\underline{x}_c, t_j) \right] \varepsilon \mathbf{R}^{N_T x 3} \tag{56b}$$

$$\underline{\hat{t}}_c^j = \left\{ \hat{t}_i(\underline{x}_c, t_j) \right\} \varepsilon \mathbf{R}^{N_T} \tag{56c}$$

$$\underline{\hat{RF}}_c^j = \left\{ \sum_{m=1}^{N_{MP}} \sum_{s=1}^{j} {}' f_{ik}^{ms}(\underline{x}_c, t_j)\ \hat{\psi}_k^{ms} + \sum_{m=1}^{N_{MP}} \left(\tilde{f}_{ik}^{m}(\underline{x}_c, t_j)\ \hat{v}_{ok}^{m} + \tilde{\tilde{f}}_{ik}^{m}(\underline{x}_c, t_j)\ \hat{u}_{ok}^{m} \right) \right\} \varepsilon \mathbf{R}^{N_T} \tag{56d}$$

Note that the vectors $\underline{RG}_c^j$ and $\underline{RF}_c^j$ give the effects of the body forces and initial conditions on the displacement and traction boundary values, respectively. By combining the singular contribution term with the appropriate ${}^{*}\mathbf{DF}$ matrices one obtains:

$$\underline{\hat{t}}_c^j = \sum_{n=1}^{N_{NP}} \sum_{s=1}^{j} \mathbf{DF}_{cn}^{js}\ \underline{P}_n^s + \underline{\hat{RF}}_c^j \qquad c = 1,2,\ldots,N_{CP} \tag{57}$$

If the collocation point coincides with the p^{th} node, the relationship between $^{*}\mathbf{DF}$ and $\mathbf{DF}$ is given as:

$$\mathbf{DF}_{cn}^{js} = {}^{*}\mathbf{DF}_{cn}^{js} \pm \frac{1}{2} \mathbf{I}\, \delta_{js}\, \delta_{np}$$

Let N_{UEQ} denote the total number of equations associated with displacement boundary conditions and N_{TEQ} denote the total number of equations associated with traction boundary conditions; that is, if we indexed N_U and N_T for each collocation point we would have:

$$N_{UEQ} = \sum_{c=1}^{N_{CP}} N_U^c , \qquad N_{TEQ} = \sum_{c=1}^{N_{CP}} N_T^c$$

where $N_{UEQ} + N_{TEQ} = N_{EQ} = 3N_{NP} = 3N_{CP}$. Combining the equations associated with each collocation point into a single system of equations gives:

$$\begin{Bmatrix} \hat{\underline{u}}^j \\ \hat{\underline{t}}^j \end{Bmatrix} = \sum_{s=1}^{j-1} \begin{bmatrix} \mathbf{G}^s \\ \mathbf{F}^s \end{bmatrix} \hat{\underline{P}}^s + \begin{bmatrix} \mathbf{G}^j \\ \mathbf{F}^j \end{bmatrix} \bar{\underline{P}}^j + \hat{\underline{R}}^j \tag{58}$$

where

$$\hat{\underline{u}}^j = \begin{Bmatrix} \hat{\underline{u}}_1^j \\ \hat{\underline{u}}_2^j \\ \vdots \\ \hat{\underline{u}}_{N_{CP}}^j \end{Bmatrix} \varepsilon\, \mathbf{R}^{N_{UEQ}} , \qquad \hat{\underline{t}}^j = \begin{Bmatrix} \hat{\underline{t}}_1^j \\ \hat{\underline{t}}_2^j \\ \vdots \\ \hat{\underline{t}}_{N_{CP}}^j \end{Bmatrix} \varepsilon\, \mathbf{R}^{N_{TEQ}}$$

$$\underline{P}^s = \begin{bmatrix} \underline{P}_1^s \\ \underline{P}_2^s \\ \vdots \\ \underline{P}_{N_{NP}}^s \end{bmatrix} \varepsilon \; \mathbf{R}^{N_{EQ}}, \qquad \hat{\underline{R}}^j = \begin{bmatrix} \hat{\underline{RG}}_1^j \\ \vdots \\ \hat{\underline{RG}}_{N_{CP}}^j \\ \hat{\underline{RF}}_1^j \\ \vdots \\ \hat{\underline{RF}}_{N_{CP}}^j \end{bmatrix} \varepsilon \; \mathbf{R}^{N_{EQ}}$$

and $\mathbf{G}^s$ and $\mathbf{F}^s$ are assembled as shown for Equation 50 except that the order of the submatrices varies as indicated in Equations 55 and 56; thus, some of the submatrices do not even exist. For example, if collocation point number 3 corresponds to a point with traction boundary conditions only (i.e., N_T=3 and N_U=0) then: (1) $\mathbf{DG}_{3n}^{js}$ does not exist for all j,s,n; and (2) $\mathbf{DF}_{3n}^{js} \varepsilon \mathbf{R}^{3x3}$ for all j,s,n. So we have

$$\mathbf{G}^s \varepsilon \mathbf{R}^{N_{UEQ} x N_{EQ}} \quad \text{and} \quad \mathbf{F}^s \varepsilon \mathbf{R}^{N_{TEQ} x N_{EQ}}$$

such that the combination of these partition matrices gives square matrices. Isolating the unknowns on the left-hand side gives the following set of equations for the artificial tractions at the the end of the j^{th} time step,

$$\begin{bmatrix} \mathbf{G}^j \\ \mathbf{F}^j \end{bmatrix} \underline{\tilde{P}}^j = \begin{bmatrix} \hat{\underline{u}}^j \\ \hat{\underline{t}}^j \end{bmatrix} - \sum_{s=1}^{j-1} \begin{bmatrix} \mathbf{G}^s \\ \mathbf{F}^s \end{bmatrix} \hat{\underline{P}}^s - \hat{\underline{R}}^j \qquad (59)$$

As previously discussed, the IBEM solves for the artificial traction values instead of the unknown boundary values. With the artificial tractions determined, unknown responses on the boundary or in the domain, over the interval of the j^{th} time step, can be calculated. As an example, to calculate internal displacements an equation like Equation 53a can be written, however $\underline{x}$ is not necessarily on Γ_U; this also could easily be expressed in matrix notation.

Efficient Implementation. To use the term "efficient" in addressing the numerical implementation of the time domain boundary element method is a bit tongue-in-cheek; Equations 52 and 59 reflect the numerical burden that is characteristic of formulations which use integral equations in time -- convolution. In this section, we will see that the "bad news" of the previous two sections can be softened considerably -- but not forgotten. We will first consider a few properties of the Green's function which can be exploited and then consider how modern trends in computer architecture, parallel processing, lend themselves to convolution.

Cole, Kosloff, and Minster (1978) appear to have been the first to present the time domain DBEM for elastodynamics. The development in their initial work, for the sake of a simplified explanation, was limited to the two-dimensional case of antiplane strain (a scalar BIVP). Their work contains an excellent discussion on how the Green's function properties can be exploited to improve numerical efficiency and how these properties motivate their selection of interpolation functions. We have previously selected a class of interpolation functions and are concentrating on numerical efficiency in this section.

Initially, it appears that Equations 52 and 59 require the calculation of $O(N_{TS}^2 N_{EQ}^2)$ coefficients; or following Cole, Kosloff, and Minster's notation $O(N_{TS}^2 N_{CP}^2)$ discrete kernals, $\mathbf{DG}_{cn}^{js}$ and $\mathbf{DF}_{cn}^{js}$. However, the Green's function has the property of time translation which can be expressed as:

$$\begin{aligned} G_{ij}(\underline{x},t+\Delta t;\underline{\xi},\tau+\Delta t) &= G_{ij}(\underline{x},t;\underline{\xi},\tau) \\ F_{ij}(\underline{x},t+\Delta t;\underline{\xi},\tau+\Delta t) &= F_{ij}(\underline{x},t;\underline{\xi},\tau) \end{aligned} \tag{60}$$

If the temporal interpolation functions have the same property, that is:

$$T_{n+k}(t+k\Delta t) = T_n(t) \tag{61}$$

then one has for the discrete kernals that,

$$\begin{aligned} \mathbf{DG}_{cn}^{j+k\ s+k} &= \mathbf{DG}_{cn}^{js} \\ \mathbf{DF}_{cn}^{j+k\ s+k} &= \mathbf{DF}_{cn}^{js} \end{aligned} \tag{62}$$

For this property to be fully exploited, Equation 61 requires the analysis to be limited to uniform time steps. With uniform time steps only the full matrices relating the j^{th} time step to the 1^{st} time step must be calculated. This includes matrices resulting from the numerical approximation of both boundary and domain integrations. Exploiting this property then reduces the number of discrete kernal calculations to $O(N_{TS}N_{CP}^2)$ assuming we are able to save matrices calculated at previous time steps. For the DBEM (see Equation 52), this means at the j^{th} time step we must have saved $\mathbf{G}^s$ and $\mathbf{F}^s$ for $s=1,2,\ldots,j-1$ or $2N_{EQ}^2(j-1)$ floating point numbers. Similarly, for the IBEM (see Equation 59) $N_{EQ}^2(j-1)$ floating point numbers must be saved for a BE analysis. For a coupled solution approach, the IBEM must also save $2N_{EQ}^2(j-1)$ floating point numbers since both the interface displacements and tractions are unknown. This will be more apparent in the following section.

In a coupled solution approach adaptive adjustment of the time steps in an analysis will greatly increase the cost associated with the BE subdomain. If the time step size is only "occasionally" changed the piecewise uniformity can be exploited but to a lesser advantage. The determination of the "optimum" amount of adaptive time step adjustment is a subject for numerical parameter studies.

The time translation property also indicates the possibility of numerical instability. Cole, Kosloff, and Minster (1978) consider the iterative process as being similar to a finite difference method on the boundary where the difference molecule expands backward in time with each step. They show that when a linear interpolation in time is used for the tractions, the process is marginally stable in theory and they also indicate it has been found to be unstable in practice.

Two more properties of the Green's function allow us to reduce the number of unique coefficient calculations and associated storage. The first property is causality, which simply says "there is no response at a given point in the domain due to an impulse load until the dilational wave has had time to travel to that point." It can also be seen (Equation 22) that the response of the point is again quiescent after the shear wave passes. In general, the nonzero response in the domain due to a concentrated impulse load is bounded by two spheres traveling at

the speeds of the dialational and distortional waves, respectively. In $\Omega \times T^+$ the region affected by the impulse load is bounded by two hypercones with axes along the time axis (see e.g., Cole, Kosloff, and Minster, 1978). The causality and post shear wave quiescence can be expressed as:

$$G_{ij}(\underline{x},t;\underline{\xi},\tau) = 0 \qquad (t-\tau)C_p < r \tag{63a}$$

and

$$G_{ij}(\underline{x},t;\underline{\xi},\tau) = 0 \qquad (t-\tau)C_S > r \tag{63b}$$

respectively.

We will inherently violate both of these with the discrete kernals. As indicated by Cole et al. (1978), the use of interpolators with separable space and time dependence is noncausal. Consider the spacial linear interpolation functions of the FEM. When the wave front passes a given node the interpolation function instantaneously becomes nonzero in a localized region. Since this region can extend beyond the wave front there is a noncausal behavior in the numerical approximation; that is, our numerical approximation results in responses at points prior to the dilational waves arrival. Cole et al. (1978) indicate that they do not expect the errors due to this effect to be large if very localized interpolators are used unless wavelengths comparable to the node separation are encountered. So the motivation for using higher order elements to reduce the number of degrees of freedom is opposed by the motivation to reflect the causality property in the numerical model.

The causality property and post shear wave quiescence of the Green's function cause many of the discrete kernals to be zero if we impose restrictions on the spacial and temporal interpolation functions. Cole et al. (1978) give general requirements since they are motivating their selection of interpolation functions. The restriction is simply the local support of the interpolation functions in time and space; for our interpolation functions this is expressed as:

$$N_n(\underline{x}) = 0 \qquad |\underline{x}-\underline{x}_n| > L \tag{64a}$$

$$T_s(t) = 0 \qquad t \leq t_{s-1} \quad \text{or} \quad t > t_s \tag{64b}$$

where L is the maximum distance from $\underline{x}_n$ to any point on an element adjoining the nodal point. For interpolation functions associated with an extreme node, L is bounded by the maximum of the two adjoining element lengths. With the local support the discrete kernals reflect the causality property as:

$$\mathbf{DG}_{cn}^{js} = \mathbf{DF}_{cn}^{js} = 0 \qquad (t_j - t_{s-1})C_P < |\underline{x}_n - \underline{\xi}_c| - L \qquad (65a)$$

$$\mathbf{DG}_{cn}^{js} = \mathbf{DF}_{cn}^{js} = 0 \qquad (t_j - t_s)C_S > |\underline{x}_n - \underline{\xi}_c| + L \qquad (65b)$$

This gives a lower limit on which discrete kernals are zero; for some combinations of interpolation function and collocation points the L are not needed in the inequalities of Equation 65. These relations (Equation 65) can be thought of as the discrete analogs to the causality property and post shear wave quiescence of the Green's function.

A numerical implementation can then use inequalities similar to the above to eliminate many discrete kernal integrations and their subsequent storage. For an out of core routine (i.e., where the coefficient matrices are written to disk) one might save a Boolean variable prior to saving each discrete kernal where the Boolean variable has the value of "true" only if the discrete kernal values are nonzero. This would then only require a 1-bit read for zero discrete kernals as apposed to a 576-bit read (assuming a 64-bit real word).

Before we leave the subject of causality, we should note that Cole et al. (1978) gave a criteria for the time step as $\Delta t < L/(2C_p)$ where L was the minimum element length. They discuss the motivation for this criteria in terms of the backward causality cone (see the reference for details). Other researchers (e.g., Manolis, 1983 and Karabalis and Beskos, 1984) suggest similar criteria for three-dimensional problems.

The last properties of the Green's function which can be exploited to reduce the numerical effort are the spacial translational and rotational symmetries. The translational symmetry can be expressed as:

$$G_{ij}(\underline{x}+\underline{z},t;\underline{\xi}+\underline{z},\tau) = G_{ij}(\underline{x},t;\underline{\xi},\tau) \qquad (66)$$

In general, these symmetries are not present in the discrete kernals since the geometric descretization lacks spacial regularity; however, for a coupled problem the geometry of the interface is relatively arbitrary and thus it can be defined to exploit these symmetries. The simplest interface geometry for a dry dock might consist of a half cylinder with hemispherical ends.

An implementation which exploits these symmetries would probably be highly specialized being based on a given interface surface geometry. The analysis cost would be reduced since there would be fewer numerical integrations and associated storage; some of the integrations would be identical to previous integrations and some could be obtained by the orthogonal transformation of previous integrations. The analysis cost would be reduced but the implementation cost would be increased. Significant bookkeeping would be necessary to relate integrations to previous integration results.

The last item to consider in this section is parallel processing. The equations which arise in the numerical approximation of integral equations lend themselves to parallel calculations. Consider the forms of Equations 52 and 59. At each step the new coefficient matrices could be calculated in a parallel manner assigning each processor to specific elements. In addition, the matrix multiplications associated with the boundary values (or artificial tractions) of previous steps could be assigned to different processors. The parallel calculation of the matrix multiplications necessitates either: (1) a large amount of dedicated memory for each processor, or (2) an architecture which performs IO in a parallel manner also so the system would not be "bus limited."

In this discussion, we have tacitly assumed that the individual integrations (i.e., Equations 47, 53, and 54), are efficiently evaluated. We did not address the evaluation of these integrations. For details on these calculations see: Banerjee and Ahmad (1985); Banerjee, Ahmad, and Manolis (1986); and Ahmad and Banerjee (1988).

Assuming the BEM calculations are now efficient enough to be of practical use, we consider how to combine the BEM with the FEM in the next section. For examples of the application of the time domain BEM see: Manolis (1983 and 1984); Karabalis and Beskos (1984, 1985, and

1986); Karabalis, Spyrakos, and Beskos (1984); Mansur and Brebbia (1985); Banerjee and Ahmad (1985); Banerjee, Ahmad, and Manolis (1986); and Ahmad and Banerjee (1988).

Coupling the BEMs with the FEM

In this section we consider coupling the direct and indirect BEMs to the FEM. The coupling approach is based on an idea presented by Dr. Benjamin Loret (1987) during a seminar at the University of California, Davis. In his presentation he gave the results of an iterative coupling between the FEM and an integral equation method for a homogeneous half-plane in elastostatics. The essence of his work was the treatment of the BE subdomain as a nonlinear boundary condition to the FE subdomain. We extend the idea to time domain problems in this section. Very brief outlines of algorithms for the IBEM and DBEM are given; the FEM is not presented in any detail.

As initially indicated in this report, the motivation for coupling the two methods is to let each method model the portions of the domain for which it is best suited. In particular, we apply the BEM to the infinite domain at whatever distance we are willing to model the media as a homogeneous, isotropic, linear elastic material. The remainder of the problem, structure and soil, is modeled by the FEM which is well suited to inhomogeneous, anisotropic, inelastic materials.

The problem is in general nonlinear since the constitutive law governing the material behavior in the FE region is inelastic. We then add another nonlinearity by treating the BE region simply as a nonlinear boundary condition to the FE region. The main steps in the coupling are outlined below:

1. Assume Γ_I is fixed during the first time step. (Assumes the body is at rest at the beginning of the analysis.)

2. With the given displacement boundary conditions on Γ_I, use the FEM to calculate the generalized nodal forces along Γ_I.

3. Assuming the BE region displaces like its interpolation functions along the boundary, calculate the traction distribution on Γ_I from the generalized nodal forces.

4. With the given traction distribution at the end of the time step, calculate the corresponding displacements along Γ_I.

5. Repeat steps 2 through 4 until the nodal forces and displacements converge.

6. Repeat steps 2 through 5 for each time step using as the initial displacement estimate the interface displacement obtained at the end of the previous time step.

The temporal aspects of the FEM are handled by a finite difference approximation; the temporal aspects of the BEM are handled by a numerical approximation of the convolution integral. The coupling of the two methods iterates at each time step to approximately satisfy equilibrium and continuity at their interface.

We consider each of the BE formulations below in more detail. For brevity assume the BE subdomain's boundary consists of Γ_I alone. This is not a limitation of the coupling method.

DBEM-FEM Coupling. Assumming traction boundary conditions, Equation 52 can be written as:

$$\mathbf{F}^j \, \underline{\bar{u}}^j = \mathbf{G}^j \, \underline{\hat{t}}^j + \underline{\hat{b}} \tag{67a}$$

where

$$\underline{\hat{b}} = \sum_{s=1}^{j-1} \left(\mathbf{G}^s \, \underline{\hat{t}}^s - \mathbf{F}^s \, \underline{\hat{u}}^s \right) + \underline{\hat{R}}^j \tag{67b}$$

and $\underline{\bar{u}}^j$ and $\underline{\hat{t}}^j$ are the interface displacements and tractions, respectively.

An overview of the coupling algorithm is presented below. The algorithm is described in pseudo-code with a Pascal/Modula 2 dialect. Consistent with the mentioned languages, supplemental comments which would not represent actual code are enclosed with (* *)'s. The FE calculations are simply represented by a single call to a procedure named FE_Calculations.

```
FOR j:= 1 TO N_TS DO  (* for each time step *)
     IF time steps are uniform THEN
          FOR i:= j TO 2 BY -1 DO (* not executed for j=1 *)
               G^i := G^(i-1)
               F^i := F^(i-1)
          END_for_i
          Calculate G^1 and F^1
          IF j=1 THEN obtain LU factorization of F^1 END_if
               (* only factored once *)
     ELSE (* only worst case is shown -- change on time step size each
               step *)
          FOR i:= 1 TO j DO
               Calculate G^i and F^i
          END_for_i
          Obtain LU factorization of F^j
               (* factored each step for worst case *)
     END_if
     Let ū^j := ū^(j-1)  (* i.e., initially estimate the interface
                            displacements at t_j by known interface displacements
                            at t_(j-1) *)
     Calculate b̂ by Equation 67b (* contribution to known vector not
                                    dependent on step j *)
     REPEAT (* the nonlinear boundary condition iteration *)
          (* Perform the FE calculations based on the displacement BCs
             on Γ_I.  Note that f̂^j denote the generalized nodal forces
             on Γ_I. *)

          FE_Calculations(ū^j, f̂^j, iteration convergence criteria,...)
          Calculate the traction distribution t̂^j on Γ_I for the BE
             region by assuming nodal loads are obtained by weighting
             the tractions by the interpolation functions.
          Calculate a new interface displacement estimate by solving
             Equation 67a for ū^j.  Note that LU factorization
             of F^j was previously obtained.
     UNTIL nonlinear boundary condition iteration has converged
END_for_j.
```

IBEM-FEM Coupling. Assuming traction boundary conditions, Equation 59 can be written as:

$$\mathbf{F}^j \, \tilde{\underline{P}}^j = \hat{\underline{t}}^j + \hat{\underline{b}}_T \tag{68a}$$

where

$$\hat{\underline{b}}_T = - \sum_{s=1}^{j-1} \mathbf{F}^s \, \hat{\underline{P}}^s - \begin{bmatrix} \hat{\underline{RF}}_1^j \\ \vdots \\ \hat{\underline{RF}}_{N_{CP}}^j \end{bmatrix} \tag{68b}$$

and $\tilde{\underline{P}}^j$ and $\hat{\underline{t}}^j$ are the artificial and actual interface tractions, respectively. With the artificial tractions obtained by the above equation, the unknown displacements along the interface can be obtained by Equation 59 as:

$$\tilde{\underline{u}}^j = \mathbf{G}^j \, \hat{\underline{P}}^j + \hat{\underline{b}}_U \tag{69a}$$

where

$$\hat{\underline{b}}_U = \sum_{s=1}^{j-1} \mathbf{G}^s \, \hat{\underline{P}}^s + \begin{bmatrix} \hat{\underline{RG}}_1^j \\ \vdots \\ \hat{\underline{RG}}_{N_{CP}}^j \end{bmatrix} \tag{69b}$$

These two systems of equations can be used to couple the IBEM to the FEM analagous to the approach used above for the DBEM. An overview of the coupling algorithm is presented below.

```
FOR j:= 1 TO N_TS DO  (* for each time step *)
     IF time steps are uniform THEN
          FOR i:= j TO 2 BY -1 DO (* not executed for j=1 *)
               G^i := G^(i-1)
               F^i := F^(i-1)
          END_for_i
          Calculate G^1 and F^1
          IF j=1 THEN obtain LU factorization of F^1 END_if
               (* only factored once *)
     ELSE  (* only worst case is shown -- change on time step size each
              step *)
          FOR i:= 1 TO j DO
               Calculate G^i and F^i
          END_for_i
          Obtain LU factorization of F^j
               (* factored each step for worst case *)
     END_it
     Let ū^j := ū^(j-1) (* i.e., initially estimate the interface
                           displacements at t_j by known interface
                           displacements at t_(j-1) *)
     Calculate b̂_T and b̂_U by Equations 68b and 69b, respectively
          (* contributions to known vectors not dependent on step j *)
     REPEAT (* the nonlinear boundary condition iteration *)
          (* Perform the FE calculations based on the displacement BCs
             on Γ_I.  Note that f̂^j denote the generalized nodal forces on
             Γ_I. *)
          FE_Calculations(ū^j, f̂^j, iteration convergence criteria, ...)
          Calculate the traction distribution t̂^j on Γ_I for the BE region
               by assuming nodal loads are obtained by weighting the
               tractions by the interpolation functions.
          Calculate a new artificial traction vector estimate by solving
               Equation 68a for p̂^j
               Note that LU factorization of F^j was previously obtained
          Calculate a new interface displacement estimate, ū^j, by
               Equation 69a
     UNTIL nonlinear boundary condition iteration has converged .
END_for_j.
```

Note that the algorithms for coupling the direct and indirect BEMs are very similar. For problems with zero body forces and a quiescent past the two coupled approaches require the same amount of numerical effort; when body forces or nonzero intial conditions exist the direct method requires less computational effort. However, if many responses within the BE region are obtained in an analysis the IBEM could require less computational effort since it integrates the effect of a single time varying vector (the artificial tractions) instead of two vectors (the boundary displacements and tractions) like the DBEM.

Note that the above algorithms do not explicitly show:

- exploitation of causality property and post shear wave quiescence
- details of how to exploit the time translation property when the time step size changes only intermittently
- details of FE calculations
- internal response calculations for the BE region
- parallel calculation
- convergence criterias

As previously mentioned, the displacements along the boundary of the BE region do not agree with the interpolation functions except at the collocation points. Thus, the use of the interpolation functions as shape functions, which they are not, to calculate the traction distribution along the interface from the generalized nodal forces is inherently in error; it also implies that we only satisfy continuity at the collocation points. The inconsistency between the assumed and actual distribution of the displacements could possibly be accounted for with a "weighting approach" by calculating the displacement at boundary points between collocation points. At worst (assuming we have convergence) the inconsistency is a good indicator of when mesh refinement is necessary.

In the above algorithms we explicitly showed the FEM calculations within the iterative loop for the nonlinear boundary conditions (BCs). The FEM calculations themselves contain iterations to satisfy the constitutive relations (among other things). Whether the constitutive relations should be iterated to convergence within each BC iteration is a subject for numerical studies. We are uncertain of the effects on numerical efficiency and convergence. The BEM calculations in each BC iteration are of $O(N_{EQ}^2)$ (assuming the BE region is not subdivided into smaller subregions); thus, in the above algorithms we sought to minimize the number of BC iterations. It might be most effective to have the convergence criteria of the FE calculations become tighter as the BC iterations converge.

CONCLUSIONS

The time domain boundary element methods based upon the Stokes solution appear to be the best suited BE formulations for the coupled solution of structural/geotechnical interaction problems that include nonlinearities and infinite domains. The direct boundary element method (DBEM) has a very elegant analytical basis which expresses the elastodynamic boundary-initial value problem in an integral equation form using the Stokes solution and the dynamic equivalent to Betti's reciprocal work theorem. The integral equations for the indirect boundary element method can be derived from those for the DBEM by considering an exterior and interior problem with a common boundary. An artificial traction along the common boundary is determined which satisfies the boundary conditions of the actual problem while displacement compatability of the two problems is enforced. The analytical basis for both methods involves integral equations in both time and space.

While the integral equations are very elegant theoretical formulations, their numerical approximations are computationally very demanding. For more efficient computations one can exploit certain properties of the Green's function and modern computer hardware, such as parallel

processing; this requires considerable algorithm implementation effort but appears to be a necessity for anything but an academic computer program.

Time domain BEMs based upon Stokes's solution provide a very rigorous solution to the radiation boundary condition problem associated with infinite domains. The theoretical and numerical background for the methods provides a good basis for future implementation work. As to whether the coupled solution approach will provide a cost-effective solution for problems with infinite domains, speculation is a poor substitute for numerical experience. However, careful algorithm design and computer hardware advances will improve the potential of applying the methods to large structural/geotechnical problems.

RECOMMENDATIONS

The following outlines the recommended steps for the development of a coupled FEM/BEM computer code for the dynamic analysis of soil-structure interaction problems:

1. Investigate the numerical integrations necessary for the BEM in elastodynamics. That is: (a) classify the different integrations (e.g., singular versus nonsingular), (b) determine appropriate numerical integration schemes, and (c) investigate the accuracy of the numerical schemes.

2. Develop a simple stand-alone BEM computer program for elastodynamics.

3. Extend the BEM program to exploit Green's function properties.

4. Extend the BEM program to exploit parallel processing hardware.

5. Extend the BEM program for use in a coupled FEM-BEM solution.

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NOMENCLATURE

A partial list of symbols used in this report follows.

Mathematical Symbols

$$[\phi * \chi](\underline{x},t) = \begin{cases} 0 & (\underline{x},t) \in \Omega \times T^- \\ \int_0^t \phi(\underline{x},t-\tau)\, \chi(\underline{x},\tau)\, d\tau, & (\underline{x},t) \in \Omega \times T^+ \end{cases}$$

where ϕ and χ are such that $\{\phi(\underline{x},t),\chi(\underline{x},t)\} \in C^{0,0}(\Omega \times T^+)$.

This defines the Riemann Convolution. For brevity it is often denoted as simply $\phi * \chi$.

Script Symbols

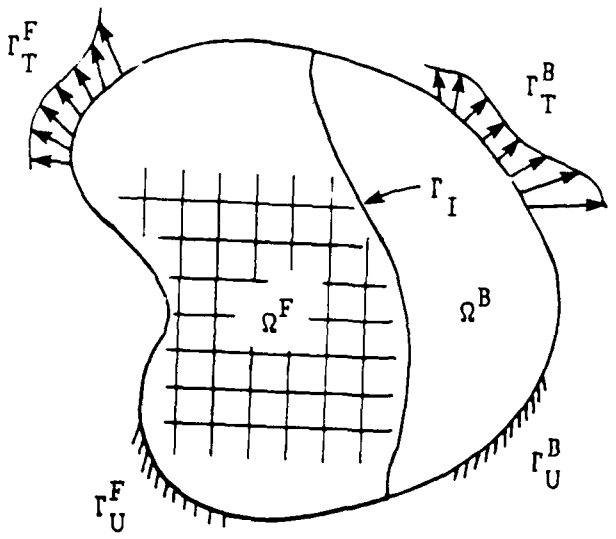
$\mathcal{E}$	Class of all elastodynamic states.
$\mathcal{E}_0$	Class of all elastodynamic states of quiescent past.
y	An elastodynamic state.

Latin Symbols

$C^{m,n}(\Omega \times T)$	Defines the class of all functions which have continuous spacial and temporal derivatives of order up to and including m and n, respectively, on $\Omega \times T$.
$\mathbf{E}_3$	Three-dimensional Euclidian space corresponding to $\mathbf{R}^3$.
$\mathbf{E}_{3\xi}$	Three-dimensional Euclidian space $\mathbf{E}_3$ omitting the point ξ.
$\mathbf{R}^n$	The linear space of ordered n-tuples of real numbers.
T^+	The time interval $[0,\infty)$. Where [and) denote the closed and open ends of the interval, respectively.
T^-	The time interval $(-\infty,0]$.
T^∞	$= T^+ \cup T^-$

Greek Symbols

δ_{ij}	Kronecker delta symbol.
$\delta(\underline{x},\underline{z})$	Dirac 'delta function.'
ε_{ij}	Strain tensor in rectangular Cartesian system.
Γ	Complete, finite boundary of the problem.
σ_{ij}	Stress tensor in rectangular Cartesian system.
Ω	Domain of the problem.
ψ_i	Vector of body forces per unit volume.
Ψ_i	Vector of body forces per unit mass.
ρ	Mass density at a given point in the domain.
ξ_i	Position vector to a point on the boundary of the problem.



$$\Omega = \Omega^F \cup \Omega^B$$

$$\Gamma^F = \Gamma_T^F \cup \Gamma_U^F \cup \Gamma_I$$

$$\Gamma^B = \Gamma_T^B \cup \Gamma_U^B \cup \Gamma_I$$

$$\Gamma_U = \Gamma_U^F \cup \Gamma_U^B$$

$$\Gamma_T = \Gamma_T^F \cup \Gamma_T^B$$

Figure 1. Elastodynamics problem using coupled solution approach.

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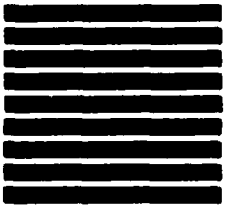

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6 Aviation Engineering Test Facilities
7 Fire prevention and control
8 Antenna technology
9 Structural analysis and design (including numerical and computer techniques)
10 Protective construction (including hardened shelters, shock and vibration studies)
11 Soil/rock mechanics
14 Airfields and pavements

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17 Expedient roads/airfields/bridges
18 Amphibious operations (including breakwaters, wave forces)
19 Over-the-Beach operations (including containerization, material transfer, lighterage and cranes)
20 POL storage, transfer and distribution

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38 Oil pollution removal and recovery
39 Air pollution

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50 Pressure vessel facilities
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52 Ocean-based concrete structures
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